

Canonical Melnikov theory for diffeomorphisms

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Abstract

We study perturbations of diffeomorphisms that have a saddle connection between a pair of normally hyperbolic invariant manifolds. We develop a first-order deformation calculus for invariant manifolds and show that a generalized Melnikov function or *Melnikov displacement* can be written in a canonical way. This function is defined to be a section of the normal bundle of the saddle connection.

We show how our definition reproduces the classical methods of Poincaré and Melnikov and specializes in methods previously used for exact symplectic and volume-preserving maps. We use the method to detect the transverse intersection of stable and unstable manifolds and relate this intersection to the set of zeros of the Melnikov displacement.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The study of the intersections of stable and unstable manifolds of maps and flows has a strong influence on dynamical systems. In particular, the existence of a transverse intersection is associated with the onset of chaos, and gave rise to the famous horseshoe construction of Smale. The Poincaré–Melnikov method [18, 34, 36] is a widely used technique for detecting such intersections. Given a system with a pair of saddles and a degenerate heteroclinic or saddle connection between them, the classical Melnikov function computes the rate at which the distance between the manifolds changes with a perturbation.

There have been many formulations of the Melnikov method for two-dimensional maps or flows [8, 11, 14, 24] and for higher-dimensional symplectic mappings [2, 4, 9]. Recently, the geometric content of Melnikov's method was exploited in order to detect heteroclinic intersections of Lagrangian manifolds for the case of perturbed Hamiltonian flows [38]. Here it was shown that the heteroclinic orbits are in correspondence with the zeros of a geometric object, the so-called Melnikov one-form.

For maps, the Melnikov function is an infinite sum whose domain is a saddle connection between two hyperbolic invariant sets. As usual, a simple zero of this function corresponds to a transverse intersection of stable and unstable manifolds of a perturbation of the original map.

Melnikov's method can also be used to compute transport fluxes. In particular, a *resonance zone* for a two-dimensional mapping is a region bounded by alternating segments of stable and unstable manifolds that are joined at primary intersection points [12, 30]. Because the intersection points are primary, a resonance zone is bounded by a Jordan curve and has exit and entry sets [13]. The images of these sets completely define the transport properties of the resonance zone. Moreover, the integral of the Melnikov function between two neighbouring primary intersection points is the first order approximation to the geometric flux escaping from the resonance zone [19, 29].

The method has also been applied to the case of periodically time-dependent, volume-preserving flows [35] and more generally to volume-preserving maps with fixed points [25] and invariant circles [27]. Volume-preserving maps provide perhaps the simplest, natural generalization of the class of area-preserving maps to higher dimensions. Moreover, they naturally arise in applications as the time-one Poincaré map of incompressible flows—even when the vector field of the flow is non-autonomous. Thus the study of the dynamics of volume-preserving maps has application both to fluids and magnetic fields.

Our goal in this paper is to develop, based on the theory of deformations, a general, geometrical description of the Melnikov displacement and to compare our theory with classical results. Deformation theory was first introduced in the theory of singularities [40], but was soon used in the contexts of volume and symplectic geometry. Its application to dynamical systems in [7, 21, 22] provides results that are close to our goals.

Let f_ϵ be a smooth family of diffeomorphisms such that the unperturbed map f_0 has a saddle connection Σ between a pair of compact r -normally hyperbolic invariant manifolds. Let $\nu(\Sigma) \equiv T_\Sigma M / T\Sigma$ be the algebraic normal bundle of the saddle connection. We show that there exists a canonical C^{r-1} section $\mathcal{D} : \Sigma \rightarrow \nu(\Sigma)$, called the *Melnikov displacement*, that measures the splitting of the saddle connection in first-order. We will prove that the Melnikov displacement is given by the absolutely convergent series

$$\mathcal{D} = \sum_{k \in \mathbb{Z}} (f_0^*)^k \mathcal{F}_0 = \sum_{k \in \mathbb{Z}} (f_{0*})^k \mathcal{F}_0,$$

where \mathcal{F}_ϵ is the vector field defined by $\frac{\partial}{\partial \epsilon} f_\epsilon = \mathcal{F}_\epsilon \circ f_\epsilon$.

These sums do not converge in the tangent space $T_\Sigma M$, but only in the algebraic normal bundle $\nu(\Sigma)$. The use of the algebraic normal bundle in the study of normally hyperbolic manifolds goes back to [17].

In addition, we will also show that the Melnikov displacement has a number of geometric properties. The main result in this direction is that any change of coordinates acts on the displacement by its pullback. This result will be used to obtain the natural action of any symmetries, reversing symmetries or integrals of the dynamical system on the displacement. Similarly, if the map preserves a symplectic or volume form, this gives additional structure to the displacement. For example, if f_ϵ is a family of exact symplectic maps and the normally hyperbolic invariant manifolds are fixed points, then we will show that there exists a function

$L : \Sigma \rightarrow \mathbb{R}$, the *Melnikov potential*, such that $dL = i(\mathcal{D})\omega$, where ω is the symplectic two-form. This relation is reminiscent of the definition of globally Hamiltonian vector fields. When the normally hyperbolic invariant manifolds are not fixed points (or isolated periodic points), their stable and unstable manifolds are coisotropic, but not isotropic, and so the relation $dL = i(\mathcal{D})\omega$ makes no sense.

We complete this introduction with a note on the organization of this paper. The general theory is developed in section 2. In section 3, we show how our theory reproduces the classical methods of Poincaré and Melnikov. The study of exact symplectic maps and volume-preserving maps is contained in sections 4 and 5, respectively.

2. Melnikov displacement

2.1. Deformation calculus

In this subsection we present the deformation calculus for families of diffeomorphisms and submanifolds. We shall begin with diffeomorphisms by defining a vector field associated with a deformation. Next, we construct a vector field for the deformation of (immersed) submanifolds. Finally, we will combine these results to define and compute the Melnikov displacement.

In this paper, we consider smooth families of diffeomorphisms $f_\epsilon : M \rightarrow M$, where M is an n -dimensional smooth manifold. Here, the term smooth family means that $f_\epsilon(\xi) \equiv f(\xi, \epsilon)$ is C^∞ in both variables. The map f_0 will be called ‘unperturbed’.

Definition 1 (Generating vector field [22]). *The generating vector field of a smooth family of diffeomorphisms f_ϵ is the unique vector field \mathcal{F}_ϵ such that*

$$\frac{\partial}{\partial \epsilon} f_\epsilon = \mathcal{F}_\epsilon \circ f_\epsilon. \tag{1}$$

If we regard \mathcal{F}_ϵ as a non-autonomous vector field with time ϵ , then the function $\Phi_{t,s} = f_t \circ f_s^{-1}$ represents its time-dependent flow as in [1, theorem 2.2.23]. Indeed, one has $\Phi_{s,s}(m) = m$ and

$$\frac{\partial}{\partial t} \Phi_{t,s}(m) = \frac{\partial}{\partial t} f_t(f_s^{-1}(m)) = \mathcal{F}_t(f_t(f_s^{-1}(m))) = \mathcal{F}_t(\Phi_{t,s}(m))$$

for t, s and all $m \in M$. In addition, we have the property: $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$.

Consequently, if f_ϵ is volume preserving, then \mathcal{F}_ϵ has zero divergence [1, theorem 2.2.24], and if f_ϵ is (exact) symplectic, then \mathcal{F}_ϵ is (globally) Hamiltonian. These geometric equivalences form the basis of the deformation calculus.

Remark 1. The generating vector field \mathcal{F}_ϵ was also called the *perturbation vector field* in [25]; however, we adopt the older terminology here. Sometimes, we will also refer to \mathcal{F}_ϵ as the *generator* of the family f_ϵ .

We will always use the convention that, given a family of diffeomorphisms denoted by italic letters f_ϵ , its generator is denoted by the same letter in calligraphic capitals. Recall that the pullback f^* and push-forward f_* of a diffeomorphism $f : M \rightarrow M$ act on a vector field $\mathcal{X} : M \rightarrow TM$ as follows:

$$\begin{aligned} f^*\mathcal{X} &= Df^{-1} \circ f\mathcal{X} \circ f = (Df^{-1}\mathcal{X}) \circ f, \\ f_*\mathcal{X} &= Df \circ f^{-1}\mathcal{X} \circ f^{-1} = (Df\mathcal{X}) \circ f^{-1}. \end{aligned}$$

We note that $f_* = (f^*)^{-1} = (f^{-1})^*$.

Next, our goal is to develop a first-order deformation calculus for invariant manifolds of smooth diffeomorphisms. We are mainly interested in stable and unstable invariant manifolds of r -normally hyperbolic manifolds for some $r \geq 1$, which unfortunately are just immersed submanifolds (not embedded submanifolds), and C^r (not C^∞). This gives rise to a few technicalities. Recall that a map $g : N \rightarrow M$ is an *immersion* when its differential has maximal rank everywhere. If g is one-to-one onto its image, then $W = g(N)$ is an immersed submanifold of the same dimension as N . We will denote an immersion by $W = g(N) \hookrightarrow M$ or simply $W \hookrightarrow M$ when the immersion g does not matter. For brevity, we will sometimes omit the term ‘immersed’.

Example 1. If W is the stable (respectively, unstable) invariant manifold of a fixed point of a diffeomorphism, then $N = \mathbb{R}^s$ (respectively, $N = \mathbb{R}^u$), where s and u are the number of stable and unstable directions at the hyperbolic point. When the fixed point is hyperbolic, $n = s + u$. Stable and unstable manifolds are typically not embedded because they can have points of accumulation. In this case the immersion is not a proper map.

We consider families of submanifolds of the form $W_\epsilon = g_\epsilon(N) \hookrightarrow M$, where $g(\xi, \epsilon)$ is C^r in both variables, for some $r \geq 1$. All the elements of such a family are diffeomorphic (as immersed submanifolds), because they are diffeomorphic to the same ‘base’ manifold N . Just as for f_0 , the unperturbed submanifold is denoted by W_0 .

Definition 2 (Adapted deformation). If $W_\epsilon \hookrightarrow M$ is a C^r family of immersed submanifolds, a family of diffeomorphisms $\phi_\epsilon : W_0 \rightarrow W_\epsilon$ is an *adapted deformation* when $\phi_0 = \text{Id}_{W_0}$ and $\phi(\xi, \epsilon)$ is C^r in both variables.

Adapted deformations exist since it suffices to take $\phi_\epsilon = g_\epsilon \circ g_0^{-1}$. While there is quite a bit of freedom in the choice of ϕ_ϵ , only its normal component is relevant, since this measures the actual motion of W_ϵ with ϵ . The normal component will be defined using the algebraic normal bundle. For an immersed submanifold $W \hookrightarrow M$, this is defined as the set of equivalence classes

$$\nu(W) \equiv T_W M / T W. \quad (2)$$

When M is Riemannian, this normal bundle is isomorphic to the more familiar geometric normal bundle, $T W^\perp$ (cf [16, p 96]). In general ν is a manifold of dimension n and is defined independently of any inner product structure on $T M$.

Definition 3 (Displacement vector field). The *displacement vector field* of a C^r family of immersed submanifolds $W_\epsilon \hookrightarrow M$ is the C^{r-1} section

$$\mathcal{D} : W_0 \rightarrow \nu(W_0), \quad \mathcal{D}(\xi) \equiv \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \phi_\epsilon(\xi) + T_\xi W_0, \quad (3)$$

where $\phi_\epsilon : W_0 \rightarrow W_\epsilon$ is any adapted deformation.

The displacement vector field is well defined; that is, its definition is independent of the choice of the adapted deformation, as is shown in the following lemma.

Lemma 1. Let $\phi_\epsilon, \tilde{\phi}_\epsilon : W_0 \rightarrow W_\epsilon$ be two adapted deformations. Then

$$\left[\left. \frac{\partial}{\partial \epsilon} \tilde{\phi}_\epsilon(\xi) - \frac{\partial}{\partial \epsilon} \phi_\epsilon(\xi) \right]_{\epsilon=0} \in T_\xi W_0, \quad \forall \xi \in W_0.$$

Proof. For each fixed $\xi \in W_0$, the map $\epsilon \mapsto c(\epsilon) \equiv \phi_\epsilon^{-1}(\tilde{\phi}_\epsilon(\xi))$ describes a C^r curve in W_0 such that $c(0) = \xi$ and $c'(0) \in T_\xi W_0$. Using the fact that $\tilde{\phi}_\epsilon(\xi) = \phi_\epsilon(c(\epsilon))$, we then have

$$\left[\frac{\partial}{\partial \epsilon} \tilde{\phi}_\epsilon(\xi) - \frac{\partial}{\partial \epsilon} \phi_\epsilon(\xi) \right]_{\epsilon=0} = D\phi_0(\xi)c'(0) = c'(0) \in T_\xi W_0,$$

because $D\phi_0(\xi) = \text{Id}_{T_\xi W_0}$. □

When the submanifold is invariant under a diffeomorphism, its deformations are related by means of a fundamental iterative relationship between the generating vector field of the family of diffeomorphisms and the displacement vector field of the family of submanifolds.

Proposition 2. *Let f_ϵ be a smooth family of diffeomorphisms, and $W_\epsilon \hookrightarrow M$ be a C^r family of immersed submanifolds that are invariant under f_ϵ . Then*

$$f_0^* \mathcal{D} - \mathcal{D} = f_0^* \mathcal{F}_0 \tag{4}$$

on the unperturbed submanifold W_0 , where \mathcal{D} is the displacement vector field (3).

Proof. The tangent space TW_0 is invariant under the pullback f_0^* , so the term $f_0^* \mathcal{D}$ is well defined as a section of the normal bundle $\nu(W_0)$. If $\phi_\epsilon : W_0 \rightarrow W_\epsilon$ is any adapted deformation, then $\tilde{\phi}_\epsilon \equiv f_\epsilon \circ \phi_\epsilon \circ f_0^{-1}$ is as well. Differentiating $\tilde{\phi}_\epsilon \circ f_0 = f_\epsilon \circ \phi_\epsilon$ with respect to ϵ yields

$$\frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \tilde{\phi}_\epsilon \circ f_0 = \mathcal{F}_0 \circ f_0 + Df_0 \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \phi_\epsilon,$$

where we used (1). However, by lemma 1 displacement (3) is independent of the adapted deformation, so

$$\mathcal{D} \circ f_0 = \mathcal{F}_0 \circ f_0 + Df_0 \mathcal{D}.$$

Applying $Df_0^{-1} \circ f_0$ to both sides finishes the proof. □

Identity (4) is equivalent to $\mathcal{D} = (f_0)_* \mathcal{D} + \mathcal{F}_0$. Thus, we can work either with pushforwards or pullbacks. To obtain the Melnikov displacement we will iterate these identities on the stable and unstable manifolds of a family of diffeomorphisms.

2.2. Normally hyperbolic invariant manifolds and saddle connections

The Melnikov displacement will be defined for a saddle connection between a pair of normally hyperbolic invariant manifolds. In this section we recall the definitions of these objects. There are many slightly different definitions of normally hyperbolic manifolds, see [17]. In this paper, we adopt the following.

Definition 4 (Normally hyperbolic invariant manifold). *Let $A \subset M$ be a submanifold invariant under a smooth diffeomorphism $f : M \rightarrow M$. We say that A is r -normally hyperbolic when there exist a Riemann structure on TM , a constant $\lambda \in (0, 1)$ and a continuous invariant splitting*

$$T_a M = E_a^s \oplus E_a^u \oplus T_a A, \quad \forall a \in A \tag{5}$$

such that if $L_a^{s,u} : E_a^{s,u} \rightarrow E_{f(a)}^{s,u}$ and $L_a^c : T_a A \rightarrow T_{f(a)} A$ are the canonical restrictions of the linear map $Df(a) : T_a M \rightarrow T_{f(a)} M$ associated with the splitting (5), then

- (i) $|(L_a^c)^{-1}|^l |L_a^s| < \lambda$ and
- (ii) $|L_a^c|^l |(L_a^u)^{-1}| < \lambda$

for all $l = 0, 1, \dots, r$, and for all $a \in A$.

As is usual in the literature, the term *normally hyperbolic* will be taken to mean *1-normally hyperbolic*. Note that setting $l = 0$ in the previous definition implies that the linearizations L^s and L^u of f restricted to the stable and unstable spaces of A have the uniform bounds

$$|L_a^s|, |(L_a^u)^{-1}| < \lambda < 1, \quad \forall a \in A. \tag{6}$$

In this paper we will assume that each r -normally hyperbolic invariant manifold A is compact, although it would be sufficient that our diffeomorphisms be uniformly C^r in some neighbourhood of A . We will also assume, without loss of generality, that A is connected. One consequence is that the sets

$$W^s = W^s(A) = W^s(A, f) = \left\{ \xi \in M : \lim_{k \rightarrow +\infty} \text{dist}(f^k(\xi), A) = 0 \right\}$$

and

$$W^u = W^u(A) = W^u(A, f) = \left\{ \xi \in M : \lim_{k \rightarrow -\infty} \text{dist}(f^k(\xi), A) = 0 \right\}$$

are C^r immersed submanifolds of M that are tangent at A to $TA \oplus E^{s,u}$, see [17]. In particular, $T_A W^s \cap T_A W^u = TA$. Moreover, A and its stable and unstable invariant manifolds are persistent: given any smooth family of diffeomorphisms f_ϵ such that $f = f_0$, then for each small enough ϵ there exists a nearby r -normally hyperbolic invariant manifold A_ϵ with C^r families of immersed submanifolds $W_\epsilon^{s,u} = W^{s,u}(A_\epsilon, f_\epsilon)$.

To compute the Melnikov displacement, we will need to show that certain series are geometrically convergent; the following lemma is a key component in this proof.

Lemma 3. *Let $f : M \rightarrow M$ be a diffeomorphism with a compact normally hyperbolic invariant manifold A , and fix a point $\xi \in W^s = W^s(A, f)$. Then given any splitting $T_\xi M = T_\xi W^s \oplus N_\xi$, there exists a constant $\mu \in (0, 1)$ and an integer $n_0 > 0$ such that*

$$|(L_n^N)^{-1}| < \mu, \quad \forall n \geq n_0,$$

where $L_n^N : N_{\xi_n} \rightarrow N_{\xi_{n+1}}$ are the restrictions of $Df(\xi_n) : T_{\xi_n} M \rightarrow T_{\xi_{n+1}} M$ to the subspaces $N_{\xi_n} = Df^n(\xi)[N_\xi]$, $\xi_n = f^n(\xi)$. A similar bound holds for W^u .

Proof. Let $a \in A$ be a point such that $\lim_{n \rightarrow +\infty} \text{dist}(\xi_n, a_n) = 0$, where $a_n = f_0^n(a)$. According to the λ -lemma for normally hyperbolic manifolds [5], the complementary subspaces N_{ξ_n} tend to the unstable subspaces $E_{a_n}^u$, as $n \rightarrow +\infty$. Thus, the maps $L_n^N : N_{\xi_n} \rightarrow N_{\xi_{n+1}}$ tend to the unstable restrictions $L_{a_n}^u : E_{a_n}^u \rightarrow E_{a_{n+1}}^u$ as $n \rightarrow +\infty$, and the lemma follows from (6). It suffices to take any $\mu \in (\lambda, 1)$. \square

The Melnikov displacement will be defined as a function on the normal bundle of a *saddle connection*, which is defined as follows.

Definition 5 (Saddle connection). *Let $f : M \rightarrow M$ be a diffeomorphism with a pair of compact normally hyperbolic invariant manifolds A and B . A saddle connection between A and B is an invariant submanifold $\Sigma \subset W^u(A) \cap W^s(B)$ such that*

$$T_\xi \Sigma = T_\xi W^u(A) = T_\xi W^s(B)$$

for all $\xi \in \Sigma$.

Remark 2. The coincidence of the tangent spaces is needed in order that the manifolds have the same algebraic normal bundles on the saddle connection:

$$\nu(\Sigma) = \nu(W^u(A))|_\Sigma = \nu(W^s(B))|_\Sigma$$

since $T_\xi M / T_\xi \Sigma = T_\xi M / T_\xi W^u(A) = T_\xi M / T_\xi W^s(B)$, for all $\xi \in \Sigma$.

By definition, $\dim \Sigma = \dim W^u(A) = \dim W^s(B)$, and the manifolds A and B are not part of the saddle connection. The simplest (and most common) saddle connections are of the form $\Sigma = W^u(A) \setminus A = W^s(B) \setminus B$. In this case, we say that the unperturbed invariant manifolds are *completely doubled*. Many Melnikov problems studied in the literature fall into this category. Nevertheless, in some problems there may exist points $\xi \in W^u(A) \cap W^s(B)$, $\xi \notin A \cup B$, such that $T_\xi W^u(A) \neq T_\xi W^s(B)$, see [9]. In that case the saddle connection is strictly contained in the intersection of the stable and unstable invariant manifolds: $\Sigma \subsetneq (W^u(A) \cap W^s(B)) \setminus (A \cup B)$.

2.3. Displacement vector fields of stable and unstable invariant manifolds

In this subsection we use the fundamental iterative equation (4) to obtain infinite series for the displacements of the stable and unstable manifolds. These series are absolutely convergent, but only, as we must stress, when they are evaluated on their corresponding normal bundles. Indeed, the tangential components of these series can be unbounded. Consequently, in order to compute these sums, each term must be projected onto the normal bundle. An example will be given in section 2.5.

The proof of the following proposition is inspired by a proof given in [2], the main difference is that our setting is more geometric.

Proposition 4. *Let f_ϵ be a smooth family of diffeomorphisms such that the unperturbed map f_0 has a compact normally hyperbolic invariant manifold A_0 with stable and unstable invariant manifolds $W_0^{s,u}$. Then the displacement vector fields $\mathcal{D}^s : W_0^s \rightarrow \nu(W_0^s)$ and $\mathcal{D}^u : W_0^u \rightarrow \nu(W_0^u)$ of the families of perturbed stable and unstable invariant manifolds are given by the absolutely convergent series*

$$\mathcal{D}^s = - \sum_{k \geq 1} (f_0^*)^k \mathcal{F}_0, \quad \mathcal{D}^u = \sum_{k \leq 0} (f_0^*)^k \mathcal{F}_0.$$

Proof. We prove the claim about the stable displacement \mathcal{D}^s ; the unstable result is obtained analogously. Repeatedly applying the iterative formula (4) yields

$$\mathcal{D}^s = (f_0^*)^n \mathcal{D}^s - \sum_{k=1}^n (f_0^*)^k \mathcal{F}_0$$

for any integer $n \geq 1$. Therefore, it suffices to check that the term $(f_0^*)^n \mathcal{D}^s$ tends geometrically to zero on the normal bundle of the unperturbed stable manifold. For any point $\xi \in W_0^s = W^s(A_0, f_0)$, let $\xi_n \equiv f_0^n(\xi)$, and $\mathcal{D}_n^s \equiv \mathcal{D}^s(\xi_n) \in T_{\xi_n} M$. Then $(f_0^*)^n \mathcal{D}^s(\xi) = (\bar{L}_n)^{-1} \mathcal{D}_n^s$, where $\bar{L}_n \equiv Df_0^n(\xi) : T_\xi M \rightarrow T_{\xi_n} M$. We must show that $(\bar{L}_n)^{-1} \mathcal{D}_n^s$ tends geometrically to zero as an element of the quotient space $T_\xi M / T_\xi W_0^s$.

Given any splitting $T_\xi M = T_\xi W_0^s \oplus N_\xi$, let $\Pi_n^N : T_{\xi_n} M \rightarrow N_{\xi_n}$ be the projections onto the linear subspaces $N_{\xi_n} = Df_0^n(\xi)[N_\xi]$, and let $L_n^N : N_{\xi_n} \rightarrow N_{\xi_{n+1}}$ be the corresponding restrictions of the linear maps $L_n \equiv Df_0(\xi_n) : T_{\xi_n} M \rightarrow T_{\xi_{n+1}} M$. Then,

$$\begin{aligned} \Pi_0^N (\bar{L}_n)^{-1} \mathcal{D}_n^s &= \Pi_0^N (L_0)^{-1} (L_1)^{-1} \cdots (L_{n-1})^{-1} \mathcal{D}_n^s \\ &= (L_0^N)^{-1} (L_1^N)^{-1} \cdots (L_{n-1}^N)^{-1} \Pi_n^N \mathcal{D}_n^s \end{aligned}$$

because $\bar{L}_n = L_{n-1} \cdots L_1 L_0$ and $L_k^N \circ \Pi_k^N = \Pi_{k+1}^N \circ L_k$. Let n_0 be the integer referred to in lemma 3, and define $l_{n_0} = \prod_{k=0}^{n_0-1} |(L_k^N)^{-1}|$. Lemma 3 gives the bound

$$|\Pi_0^N (\bar{L}_n)^{-1} \mathcal{D}_n^s| \leq l_{n_0} \mu^{n-n_0} |\Pi_n^N \mathcal{D}_n^s| \leq l_{n_0} \mu^{n-n_0} |\mathcal{D}_n^s|$$

for some $\mu \in (0, 1)$. Moreover, the sequence $(\mathcal{D}_n^s)_{n \geq 0}$ is bounded due to the compactness of the normally hyperbolic manifold A_0 and the continuity of the displacement vector field $\mathcal{D}^s(\xi)$.

Therefore, $|\Pi_0^N (\bar{L}_n)^{-1} \mathcal{D}_n^s|$ tends geometrically to zero as $n \rightarrow 0$, and hence, $(\bar{L}_n)^{-1} \mathcal{D}_n^s$ tends geometrically to zero in the quotient space $T_\xi M / T_\xi W_0^s$. \square

2.4. Melnikov displacement

We will now use the displacement vector fields to study the splitting of a saddle connection upon perturbation. As usual, f_ϵ denotes a smooth family of diffeomorphisms such that the unperturbed map f_0 has a saddle connection $\Sigma \subset W^u(A_0, f_0) \cap W^s(B_0, f_0)$ between a pair of compact r -normally hyperbolic invariant manifolds A_0 and B_0 . These manifolds persist and remain r -normally hyperbolic for small ϵ .

We want to study the distance between the perturbed manifolds $W_\epsilon^u = W^u(A_\epsilon, f_\epsilon)$ and $W_\epsilon^s = W^s(B_\epsilon, f_\epsilon)$. The growth rate of this distance with ϵ is obtained simply by taking the difference between the displacement vector fields of both families.

Definition 6 (Melnikov displacement). *Under the previous assumptions, the Melnikov displacement is the canonical C^{r-1} section of the normal bundle $\nu(\Sigma)$ defined by*

$$\mathcal{D} \equiv \mathcal{D}^u - \mathcal{D}^s : \Sigma \rightarrow \nu(\Sigma), \tag{7}$$

where $\mathcal{D}^u : W_0^u \rightarrow \nu(W_0^u)$ and $\mathcal{D}^s : W_0^s \rightarrow \nu(W_0^s)$ are the displacement vector fields of W_ϵ^u and W_ϵ^s , respectively.

Remark 3. The Melnikov displacement \mathcal{D} makes sense only on the saddle connection, where the tangent spaces of the unperturbed invariant manifolds W_0^u and W_0^s coincide. Away from Σ , the difference $\mathcal{D}^u - \mathcal{D}^s$ is undefined because each term is a section of a different (algebraic) normal bundle, see remark 2.

Corollary 5. *Let $f_\epsilon : M \rightarrow M$ be a smooth family of diffeomorphisms verifying the assumptions of definition 6. Then its Melnikov displacement $\mathcal{D} : \Sigma \rightarrow \nu(\Sigma)$ is given by the absolutely convergent sums*

$$\mathcal{D} = \sum_{k \in \mathbb{Z}} (f_0^*)^k \mathcal{F}_0 = \sum_{k \in \mathbb{Z}} (f_{0*})^k \mathcal{F}_0. \tag{8}$$

Proof. From proposition 4, we get that $\mathcal{D} = \mathcal{D}^u - \mathcal{D}^s = \sum_{k \in \mathbb{Z}} (f_0^*)^k \mathcal{F}_0 = \sum_{k \in \mathbb{Z}} (f_{0*})^k \mathcal{F}_0$, where the last equality follows from the identity $(f_{0*})^k = (f_0^*)^{-k}$. \square

The Melnikov displacement has been defined in a canonical way as a section of the normal bundle; as such it has strong geometric properties. We will show next that any change of variables acts as a pullback on it. This will imply a number of geometrical properties, for example, that the displacement is invariant under the pullback and the push-forward of the unperturbed map. In addition, we will see that if there exist symmetries, reversors, first integrals or saddle connections, then these have natural implications for the Melnikov displacement. These claims are the subject of proposition 6.

We recall that a diffeomorphism $f : M \rightarrow M$ is *symmetric* when there exists a diffeomorphism $s : M \rightarrow M$ such that $f \circ s = s \circ f$, and then s is called a *symmetry* of the map f . Analogously, f is *reversible* when there exists a diffeomorphism $r : M \rightarrow M$ such that $f \circ r = r \circ f^{-1}$, and then r is called a *reversor* of the map f . In many applications, r is an involution, $r^2 = \text{Id}$, though this need not be the case [15]. Finally, a function $I : M \rightarrow \mathbb{R}$ is a first integral of f when $I \circ f = I$.

Proposition 6. *Let f_ϵ be a smooth family of diffeomorphisms verifying the assumptions of definition 6 and let \mathcal{D} be its Melnikov displacement.*

- (i) *Given any smooth family of changes of variables $h_\epsilon : M \rightarrow M$, the family $\tilde{f}_\epsilon = h_\epsilon^{-1} \circ f_\epsilon \circ h_\epsilon$ also verifies the assumptions of definition 6, its Melnikov displacement $\tilde{\mathcal{D}}$ is defined on the saddle connection $\tilde{\Sigma} \equiv h_0^{-1}(\Sigma)$, and*

$$\tilde{\mathcal{D}} = h_0^* \mathcal{D}. \tag{9}$$

- (ii) *The Melnikov displacement is invariant by the pullback and the push-forward of the unperturbed map. That is,*

$$f_0^* \mathcal{D} = (f_0)_* \mathcal{D} = \mathcal{D}. \tag{10}$$

- (iii) *If f_ϵ has a smooth family of*

- (a) *symmetries $s_\epsilon : M \rightarrow M$ such that $s_0(\Sigma) = \Sigma$, then*

$$s_0^* \mathcal{D} = (s_0)_* \mathcal{D} = \mathcal{D}. \tag{11}$$

- (b) *Reversors $r_\epsilon : M \rightarrow M$ such that $r_0(\Sigma) = \Sigma$, then*

$$r_0^* \mathcal{D} = (r_0)_* \mathcal{D} = -\mathcal{D}. \tag{12}$$

If, in addition, r_0 is an involution and its fixed set $R_0 \equiv \{\xi \in M : r_0(\xi) = \xi\}$ is a submanifold that intersects Σ transversely at some point ξ_0 , then $\mathcal{D}(\xi_0) = 0$.

- (c) *First integrals $I_\epsilon : M \rightarrow \mathbb{R}$ such that $A_\epsilon \cup B_\epsilon \subset I_\epsilon^{-1}(\{0\})$, then*

$$i(\mathcal{D})dI_0 \equiv dI_0(\mathcal{D}) = 0. \tag{13}$$

- (d) *Saddle connections $\Sigma_\epsilon \subset W_\epsilon^u \cap W_\epsilon^s$ such that $\Sigma_0 = \Sigma$, then $\mathcal{D} = 0$.*

- (iv) *If there exists a vector field $\mathcal{X} : M \rightarrow TM$ such that its flow commutes with f_ϵ , then the Lie derivative of the Melnikov displacement with respect to \mathcal{X} vanishes, that is,*

$$L_{\mathcal{X}} \mathcal{D} = 0. \tag{14}$$

It is very important to stress that the above results make sense as identities on the normal bundle of saddle connections. The relation (9) makes sense because the pullback h_0^* maps $T\Sigma$ onto $T\tilde{\Sigma}$, whereas (10) makes sense because $T\Sigma$ is invariant by the pullback f_0^* and the push-forward $(f_0)_*$. Similar arguments apply to (11) and (12). The identities (13) and (14) make sense because $T\Sigma$ is contained in the kernel of the one-form dI_0 and the vector field \mathcal{X} is tangent to Σ , respectively. The hypothesis $A_\epsilon \cup B_\epsilon \subset I_\epsilon^{-1}(\{0\})$ means that A_ϵ and B_ϵ are contained in the same level set of the first integrals, which can be assumed to be the zero level without loss of generality. This holds, for instance, in the transitive homoclinic case: $B_\epsilon = A_\epsilon$ and A_ϵ is transitive.

Proof. We can write a geometric proof based on the definition of the Melnikov displacement as a canonical section of the normal bundle, or a computational proof using the formulae (8). We follow the geometric approach.

- (i) The first claims are obvious. For the last one, note that if $\phi_\epsilon^{s,u} : W_0^{s,u} \rightarrow W_\epsilon^{s,u}$ are any adapted deformations (for the family f_ϵ), then $\tilde{\phi}_\epsilon^{s,u} \equiv h_\epsilon^{-1} \circ \phi_\epsilon^{s,u} \circ h_0$ are as well (for the family \tilde{f}_ϵ). Differentiating $h_\epsilon \circ \tilde{\phi}_\epsilon^{s,u} = \phi_\epsilon^{s,u} \circ h_0$ with respect to ϵ yields

$$\mathcal{H}_0 \circ h_0 + Dh_0 \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \tilde{\phi}_\epsilon^{s,u} = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \phi_\epsilon^{s,u} \circ h_0,$$

where we used that $\frac{\partial}{\partial \epsilon} h_\epsilon = \mathcal{H}_\epsilon \circ h_\epsilon$. Hence, $\mathcal{H}_0 \circ h_0 + Dh_0 \tilde{\mathcal{D}}^{s,u} = \mathcal{D}^{s,u} \circ h_0$, so the difference $\mathcal{D} = \mathcal{D}^u - \mathcal{D}^s$ verifies the relation $\mathcal{D} \circ h_0 = Dh_0 \tilde{\mathcal{D}}$, which is equivalent to (9).

- (ii) If we take $h_\epsilon = f_\epsilon$ in (6), then $\tilde{f}_\epsilon = f_\epsilon^{-1} \circ f_\epsilon \circ f_\epsilon = f_\epsilon$ and $\tilde{\Sigma} = f_0^{-1}(\Sigma) = \Sigma$. Therefore, $\tilde{\mathcal{D}} = \mathcal{D}$ and (10) follows from (9).
 - (iiia) If we take $h_\epsilon = s_\epsilon$ in (6), then $\tilde{f}_\epsilon = s_\epsilon^{-1} \circ f_\epsilon \circ s_\epsilon = f_\epsilon$ and $\tilde{\Sigma} = s_0^{-1}(\Sigma) = \Sigma$. Therefore, $\tilde{\mathcal{D}} = \mathcal{D}$ and (11) follows from (9).
 - (iiib) If we take $h_\epsilon = r_\epsilon$ in (6), then $\tilde{f}_\epsilon = r_\epsilon^{-1} \circ f_\epsilon \circ r_\epsilon = f_\epsilon^{-1}$ and $\tilde{\Sigma} = r_0^{-1}(\Sigma) = \Sigma$. Moreover, a stable (respectively, unstable) manifold of a map becomes unstable (respectively, stable) for the inverse map. Therefore, $\tilde{\mathcal{D}} = -\mathcal{D}$ and (12) follows from (9).
- Next, we assume that r_0 is an involution whose fixed set R_0 intersects Σ transversely at ξ_0 . That is, $r_0^2 = \text{Id}$, $r_0(\xi_0) = \xi_0$ and $T_{\xi_0}M = T_{\xi_0}\Sigma \oplus T_{\xi_0}R_0$. We want to prove that $\mathcal{D}(\xi_0) = 0$ in the normal bundle $\nu(\Sigma)$, or equivalently that $\mathcal{D}(\xi_0) \in T_{\xi_0}\Sigma$.
- Since $r_0(\xi_0) = \xi_0$ and $r_0^2 = \text{Id}$, the square of the linear endomorphism $Dr_0(\xi_0) : T_{\xi_0}M \rightarrow T_{\xi_0}M$ is the identity map. This implies that $Dr_0(\xi_0)$ is diagonalizable and its spectrum is contained in the set $\{-1, 1\}$, so $T_{\xi_0}M = E^+ \oplus E^-$, where $E^\pm = \ker(Dr_0(\xi_0) \mp \text{Id})$. We claim that $E^+ = T_{\xi_0}R_0$ and $E^- = T_{\xi_0}\Sigma$. The first claim follows from the fact that involutions are locally conjugate to their linear parts at fixed points. Since $T_{\xi_0}\Sigma$ is invariant under $Dr_0(\xi_0)$ and complementary to $T_{\xi_0}R_0 = E^+$ in $T_{\xi_0}M$, we get the second claim. Finally, the evaluation of (12) at the point $\xi = \xi_0$ yields $Dr_0(\xi_0)\mathcal{D}(\xi_0) = -\mathcal{D}(r_0(\xi_0)) = -\mathcal{D}(\xi_0)$, and so $\mathcal{D}(\xi_0) \in E^- = T_{\xi_0}\Sigma$.
- (iiic) Let $I_\epsilon = I_0 + \epsilon I_1 + O(\epsilon^2)$. Obviously, $W_\epsilon^{u,s} \subset I_\epsilon^{-1}(\{0\})$. Differentiating $I_\epsilon \circ \phi_\epsilon^{s,u} = 0$ with respect to ϵ yields $dI_0(\mathcal{D}^{s,u}) = -I_1$. Thus, $\mathcal{D} = \mathcal{D}^u - \mathcal{D}^s$ verifies (13).
 - (iiid) In this case, we can take $\phi_\epsilon^s = \phi_\epsilon^u$, and so $\mathcal{D} = \mathcal{D}^u - \mathcal{D}^s$ vanishes on $\Sigma = \Sigma_0$.
 - (iv) Let $\phi^t : M \rightarrow M$ be the flow of the vector field: $\frac{\partial}{\partial t}\phi^t = \mathcal{X} \circ \phi^t$ and $\phi^0 = \text{Id}$. Thus, given any t , the (constant) family $s_\epsilon \equiv \phi^t$ is a smooth family of symmetries of f_ϵ . Next, using relation (11), we get that $(\phi^t)^*\mathcal{D} = \mathcal{D}$ for any t , and (14) follows by definition of the Lie derivative. □

These results have been extensively used in the literature. The invariance of Melnikov objects under the unperturbed map gives rise to periodicities when suitable coordinates are used; examples can be found in [8, 14, 20]. Item (iiid) implies a simple splitting criterion: if the Melnikov displacement does not vanish identically, the separatrix splits [8, 37]. Upper bounds on the number of uniform first integrals of the family f_ϵ can be deduced from item (iiic), see [32, 41]. This result has also been used to establish necessary and sufficient conditions for uniform integrability of analytic, exact symplectic maps [21]. Symmetries have also been extensively used, for example, to improve the lower bound obtained by Morse theory for the number of critical points of some Melnikov potentials [9] (we will discuss Melnikov potentials in section 4.2.). Relation (14) is similar to Noether’s theorem, since the existence of a continuous symmetry—the flow of the vector field \mathcal{X} —implies a conservation law for the Melnikov displacement. Finally, fixed sets of reversors can be used to guarantee the existence of heteroclinic points and zeros of Melnikov functions; this is an old trick, see [10].

In the classical Melnikov method, one uses simple zeros of a Melnikov function to predict the transverse intersection of the invariant manifolds. We next show that this result also holds for the Melnikov displacement.

Theorem 7. *If ξ_0 is a simple zero of the Melnikov displacement (7), then the perturbed invariant manifolds W_ϵ^u and W_ϵ^s intersect transversely at some point $\xi_\epsilon = \xi_0 + O(\epsilon)$ for small enough ϵ .*

Proof. By definition, the saddle connection Σ is an f_0 -invariant submanifold of M . Let $\pi : \nu(\Sigma) \rightarrow \Sigma$ be the projection of the normal bundle onto Σ . There is a tubular neighbourhood N of Σ that is diffeomorphic to $\nu(\Sigma)$ via a diffeomorphism $\psi : N \rightarrow \nu(\Sigma)$, as illustrated

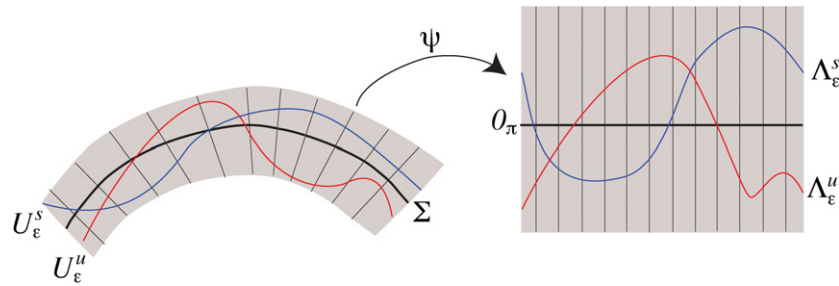


Figure 1. A tubular neighbourhood that is diffeomorphic to the normal bundle $\pi : \nu(\Sigma) \rightarrow \Sigma$.

in figure 1. Since N is an open neighbourhood of Σ in M , each deformation of Σ occurs inside N , for ϵ small enough.

We note that under the diffeomorphism ψ , and for ϵ small enough, deformations of Σ can be thought as deformations of the zero section, 0_π , in $\nu(\Sigma)$. Indeed, a deformation of Σ can be parametrized by a section $\Sigma \rightarrow \nu(\Sigma)$. Let $U_0 \subset \Sigma$ be an open set that contains ξ_0 . Let U_ϵ^s and U_ϵ^u be deformations of U_0 such that

$$U_\epsilon^s \subset W_\epsilon^s, \quad U_\epsilon^u \subset W_\epsilon^u.$$

Let $\Lambda_\epsilon^s = \psi(U_\epsilon^s)$ and $\Lambda_\epsilon^u = \psi(U_\epsilon^u)$ be the images in $\nu(\Sigma)$ of two deformations of the saddle connection, corresponding to the images of the stable and unstable manifolds. We want to show that Λ_ϵ^s and Λ_ϵ^u intersect transversely for ϵ small enough when \mathcal{D} has a simple zero.

We parametrize each manifold with a section $U_0 \rightarrow \nu(\Sigma)$. That is, if ϵ is small, then $\tilde{\phi}_\epsilon^{s,u} : U_0 \rightarrow \nu(\Sigma)$ given by

$$\tilde{\phi}_\epsilon^{s,u}(\xi) = \pi^{-1}(\xi) \cap \Lambda_\epsilon^{s,u}$$

are sections of the normal bundle restricted to U_0 . Notice that the functions $\phi_\epsilon^{s,u} : U_0 \rightarrow N \subset M$ given by

$$\phi_\epsilon^{s,u} = \psi^{-1} \circ \tilde{\phi}_\epsilon^{s,u}$$

are adapted deformations of U_0 with images $U_\epsilon^{s,u}$. In fact, both are perturbations of the zero section. From the definition of displacements we have that

$$\tilde{\phi}_\epsilon^{s,u}(\xi) = \epsilon \tilde{\mathcal{D}}^{s,u}(\xi) + O(\epsilon^2),$$

where

$$\tilde{\mathcal{D}}^{s,u}(\xi) = D\psi(\xi)\mathcal{D}^{s,u}(\xi)$$

and $\mathcal{D}^{s,u}(\xi)$ are the displacements of $\phi_\epsilon^{s,u}$. Therefore, each manifold Λ_ϵ^s and Λ_ϵ^u is the image of $\tilde{\phi}_\epsilon^s$ and $\tilde{\phi}_\epsilon^u$ and they intersect transversely at $\tilde{\phi}_\epsilon^s(\xi_0) = \tilde{\phi}_\epsilon^u(\xi_0)$ if and only if ξ_0 is a simple zero of the section $\tilde{\phi}_\epsilon^u - \tilde{\phi}_\epsilon^s$.

Now we use the standard ‘blow up’ argument so that the implicit function theorem can be applied. Let $\bar{\phi}_\epsilon^{s,u} : \Sigma \rightarrow \nu(\Sigma)$ be the section given by

$$\bar{\phi}_\epsilon^{s,u} = \begin{cases} \frac{1}{\epsilon} \tilde{\phi}_\epsilon^{s,u} & \text{for } \epsilon \neq 0, \\ \tilde{\mathcal{D}}^{s,u} & \text{for } \epsilon = 0. \end{cases}$$

Notice that $\bar{\phi}_\epsilon^{s,u}$ is C^{r-1} if $\tilde{\phi}_\epsilon^{s,u}$ is C^r , and moreover that, when $\epsilon \neq 0$, ξ_ϵ is a simple zero of $\tilde{\phi}_\epsilon^u - \tilde{\phi}_\epsilon^s$ if and only if it is for $\bar{\phi}_\epsilon^u - \bar{\phi}_\epsilon^s$. Finally, since $D\psi(\xi)\mathcal{D}(\xi) = (\bar{\phi}_0^u - \bar{\phi}_0^s)(\xi)$ and Λ_ϵ^s

and Λ_ϵ^u are the images of $\tilde{\phi}_\epsilon^s$ and $\tilde{\phi}_\epsilon^u$, the implicit function theorem implies that if ξ_0 is a simple zero of \mathcal{D} , then Λ_ϵ^s and Λ_ϵ^u intersect transversely near ξ_0 , for ϵ small enough. Thus W_ϵ^s and W_ϵ^u intersect transversely near ξ_0 , for ϵ small enough. \square

2.5. Example: perturbed Suris map

As a simple example, we consider the generalized standard map

$$f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}, \quad f(x, y) = (x + y - V'(x), y - V'(x)), \quad (15)$$

where $V : \mathbb{T} \rightarrow \mathbb{T}$ is a periodic potential. It is easy to see that f preserves area and orientation and that its fixed points have the form $(x^*, 0)$ where $V'(x^*) = 0$. Such fixed points are saddles if and only if $V''(x^*) < 0$ or $V''(x^*) > 4$, because $\text{tr}(\mathcal{D}f(x^*, 0)) = 2 - V''(x^*)$.

Following McMillan [31], we can find a generalized standard map with a saddle connection between two saddle points if we choose a diffeomorphism $c : \mathbb{R} \rightarrow \mathbb{R}$ such that $c(x + 2) = c^{-1}(x + 1) + 1 = c(x) + 2$, and let

$$V'(x) = 2x - c(x) - c^{-1}(x). \quad (16)$$

To see this, first note that, with this choice, the force $V'(x)$ is periodic with period one. Moreover, if x^* is a hyperbolic fixed point of c (that is, $c(x^*) = x^*$ and $0 < c'(x^*) \neq 1$), then

$$V''(x^*) = 2 - c'(x^*) - (c^{-1})'(x^*) = 2 - c'(x^*) - 1/c'(x^*) < 0.$$

Thus, $(x^*, 0)$ is a saddle fixed point.

Moreover, the graphs of the functions $\chi_\pm(x) \equiv x - c^{\pm 1}(x)$ are invariant and the dynamics on these sets is very simple:

$$\begin{aligned} f^k(x, \chi_-(x)) &= (c^k(x), \chi_-(c^k(x))), \\ f^k(x, \chi_+(x)) &= (c^{-k}(x), \chi_+(c^{-k}(x))) \end{aligned} \quad (17)$$

for all $k \in \mathbb{Z}$. These graphs contain saddle connections if we choose a pair of neighbouring fixed points a and b of c , so that c has no fixed points in (a, b) . Suppose further that a (respectively, b) is a stable (respectively, unstable) fixed point of c , so that $\lim_{k \rightarrow +\infty} c^k(x) = a$ and $\lim_{k \rightarrow +\infty} c^{-k}(x) = b$ for all $x \in (a, b)$. Then $A = (a, 0)$ and $B = (b, 0)$ are saddle points of the map f , and

$$\Sigma_\pm = \{(x, \chi_\pm(x)) \in \mathbb{T} \times \mathbb{R} : x \in (a, b)\}$$

are saddle connections between them: $\Sigma_- \subset W^s(A) \cap W^u(B)$ and $\Sigma_+ \subset W^u(A) \cap W^s(B)$, see figure 2.

It is known that a generalized standard map with a potential of the form (16) is typically non-integrable [27]. An integrable example, f_0 , for each $\mu \in (0, 1)$ is obtained when the diffeomorphism c is given by

$$c(x) = c_\mu(x) = \frac{2}{\pi} \arctan \left(\frac{(\mu + 1) \tan(\pi x/2) + (\mu - 1)}{(\mu - 1) \tan(\pi x/2) + (\mu + 1)} \right) \quad (18)$$

for $-1 \leq x \leq 1$. This function, when extended to \mathbb{R} using $c(x + 2) = c(x) + 2$, gives the period-one potential (16)

$$V_0(x) = \frac{2}{\pi} \int_0^x \arctan \left(\frac{\delta \sin(2\pi t)}{1 + \delta \cos(2\pi t)} \right) dt, \quad \delta = \frac{(1 - \mu)^2}{(1 + \mu)^2}. \quad (19)$$

A first integral of the map is $I(x, y) = \cos \pi y + \delta \cos \pi(2x - y)$ [26, 33, 39].

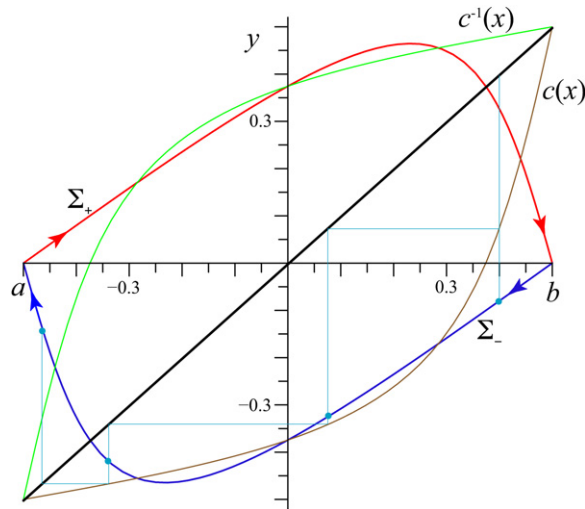


Figure 2. Saddle connections Σ_+ and Σ_- for the Suris map with potential (19) and the diffeomorphisms $x \mapsto c^{\pm 1}(x)$ for $\mu = 0.2$. Also shown are images of a point on the saddle connection Σ_- , obtained by iterating $x \mapsto c(x)$.

The diffeomorphism (18) is conjugate to a Möbius transformation, and it is easy to find an explicit formula for its iterations:

$$(c_\mu)^k(x) = c_{\mu^k}(x), \quad \forall k \in \mathbb{Z}. \tag{20}$$

The points $a = -1/2$ and $b = 1/2$ are fixed points of c . The point a is stable and b is unstable because $c'(a) = \mu \in (0, 1)$ and $c'(b) = 1/\mu > 1$.

We now perturb the integrable map by modifying the potential (19):

$$V_\epsilon(x) = V_0(x) + \epsilon U(x). \tag{21}$$

We compute the Melnikov displacement using (8). For this calculation we must first compute the vector field \mathcal{F}_0 , which for the generalized standard map (15) with potential (21) is

$$\mathcal{F}_0(x, y) = -U'(x - y)(1, 1)^T.$$

The series (8) for the displacement \mathcal{D} is easily computed by iterating f_0 and its inverse along the orbit. The relations (17) and (20) make it even easier since the iterations reduce to evaluations of c . However, in order to ensure convergence of the series, we must take into account the fact that it is only the normal component of the displacement that is desired; indeed, the iteration of the tangential component is not bounded. Let $\Pi^N : T_\Sigma M \rightarrow \nu(\Sigma)$ be the canonical projection onto the normal bundle (2). Due to the fact that Σ is invariant under f_0 , we have that $\Pi^N \circ f_0^* = f_0^* \circ \Pi^N$. To avoid numerical errors, we project each term in the sum (8). We display in figure 3 the projection $\Pi^N(\mathcal{D})$ as a function of x along Σ_- , for two perturbative potentials $U(x)$.

3. Comparison with classical methods

The Melnikov displacement (7) generalizes the classical methods used to detect the splitting of separatrices due to Poincaré and Melnikov.

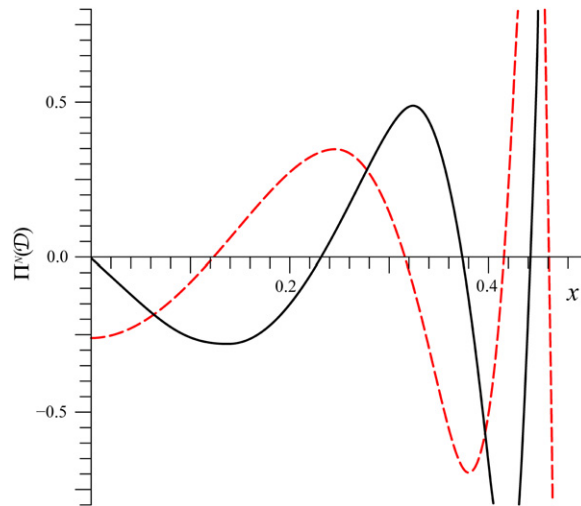


Figure 3. Normal component of \mathcal{D} for the Suris map on Σ_- with $\mu = 0.2$. The solid curve is for $U(x) = -\cos(2\pi x)/2\pi$ and the dashed curve is for $U(x) = \sin(2\pi x)/2\pi$.

3.1. Poincaré method

Poincaré’s method [36] is based on the existence of first integrals, such as an energy function, for the unperturbed system. Assume that f_0 has a saddle connection Σ between a pair of hyperbolic fixed points a and b . For the simplest case, Σ has codimension one, and a single first integral suffices: $I \circ f_0 = I$. Saddle connections with a higher codimension can be treated in a similar way if there are sufficiently many integrals. The splitting between the stable and unstable manifolds is measured by the rate of change in the first integral I with ϵ on the saddle connection. That is, we define the Poincaré function $M_I : \Sigma \rightarrow \mathbb{R}$ by

$$M_I = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} (I \circ \phi_\epsilon^u - I \circ \phi_\epsilon^s), \tag{22}$$

where ϕ_ϵ^u and ϕ_ϵ^s are deformations adapted to the perturbed invariant manifolds.

Proposition 8. *Under the above assumptions,*

$$M_I = i(\mathcal{D})dI \equiv dI(\mathcal{D}),$$

where \mathcal{D} is the Melnikov displacement (7).

The proof is a simple computation. The term $dI(\mathcal{D})$ makes sense because the first integral is constant on the saddle connection, and so the tangent space $T\Sigma$ is contained in the kernel of dI . If $dI|_\Sigma$ is non-degenerate, then a point $\xi_0 \in \Sigma$ is a simple zero of M_I if and only if it is a simple zero of \mathcal{D} . In this case, the perturbed invariant manifolds intersect transversely near ξ_0 for ϵ small enough. Therefore, the Melnikov displacement \mathcal{D} generalizes the estimates of splitting in the Poincaré style.

3.2. Melnikov method

The classical Melnikov method [34] is based on estimating the movement of a manifold in a direction normal to the separatrix. To define the normal, the phase space M is assumed to have a Riemannian inner product $\langle \cdot, \cdot \rangle$. As before, assume for simplicity that Σ is a saddle

connection of codimension one. The appropriate normal to Σ is called an adapted normal vector field.

Definition 7 (Adapted normal vector field [25]). Let Σ be a f_0 -invariant submanifold of codimension one. A vector field $\eta : \Sigma \rightarrow T_\Sigma M$ is adapted normal when

- (i) η is non-degenerate: $\eta(\xi) \neq 0$, for all $\xi \in \Sigma$,
- (ii) η is normal: $\langle \eta(\xi), v \rangle = 0$, for all $\xi \in \Sigma$ and $v \in T_\xi \Sigma$ and
- (iii) η is invariant: $f_0^* \langle \eta, \mathcal{Y} \rangle = \langle \eta, f_0^* \mathcal{Y} \rangle$, for all vector field $\mathcal{Y} : \Sigma \rightarrow T_\Sigma M$.

Using this notation, the classical Melnikov function $M_\eta : \Sigma \rightarrow \mathbb{R}$ is

$$M_\eta = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \langle \eta, \phi_\epsilon^u - \phi_\epsilon^s \rangle, \tag{23}$$

where ϕ_ϵ^u and ϕ_ϵ^s are deformations adapted to the perturbed invariant manifolds. Consequently, M_η is related to the Melnikov displacement (7) by

$$M_\eta = \langle \eta, \mathcal{D} \rangle. \tag{24}$$

Since η is non-degenerate, M_η and \mathcal{D} have the same simple zeros. We also note that (24) makes sense because η is normal, and so the component of \mathcal{D} in the tangent space $T\Sigma$ does not play any role.

Remark 4. Adapted normal vector fields can be obtained as gradients of non-degenerate first integrals. Let I be a smooth first integral of f_0 such that its gradient ∇I does not vanish on Σ . Given a Riemannian structure, the gradient is the unique vector field such that $i(\mathcal{Y})dI = \langle \nabla I, \mathcal{Y} \rangle$ for any vector field \mathcal{Y} on M . Therefore, $f_0^* \langle \nabla I, \mathcal{Y} \rangle = \langle \nabla(I \circ f_0), f_0^* \mathcal{Y} \rangle$, and $\eta = \nabla I$ is an adapted normal vector field. Obviously, the Poincaré function (22) and the Melnikov function (23) coincide when $\eta = \nabla I$.

4. Exact symplectic maps

4.1. Basic results

In this section, we will see how the Melnikov displacement (7) generalizes previous theories developed for hyperbolic fixed points of exact symplectic maps. We will also show why general normally hyperbolic invariant manifolds cannot be studied in the same way.

A $2n$ -dimensional manifold M is *exact symplectic* when it admits a non-degenerate two-form ω such that $\omega = -d\lambda$ for some Liouville one-form λ . The typical example of an exact symplectic manifold is provided by a cotangent bundle $M = T^*Q$, together with the canonical forms $\omega_0 = dx \wedge dy$ and $\lambda_0 = ydx$, in cotangent coordinates (x, y) .

A map $f : M \rightarrow M$ is *exact symplectic* if $\oint_\gamma \lambda = \oint_{f(\gamma)} \lambda$ for any closed path $\gamma \subset M$ or, equivalently, if there exists a *generating* (or *primitive*) *function* $S : M \rightarrow \mathbb{R}$ such that $f^* \lambda - \lambda = dS$. In particular, an exact symplectic map is *symplectic*: $f^* \omega = \omega$.

A submanifold N of M is *exact isotropic* if $\oint_\gamma \lambda = \oint_{f(\gamma)} \lambda$ for any closed path $\gamma \subset N$ or, equivalently, if there exists a generating function $L : N \rightarrow \mathbb{R}$ such that $j_N^* \lambda = dL$. Here, $j_N : N \hookrightarrow M$ denotes the natural inclusion map. In particular, an exact isotropic submanifold is *isotropic*: $j_N^* \omega = 0$. The maximal dimension of an isotropic submanifold is n , and when the dimension is n , the submanifold is called *Lagrangian*.

A vector field $\mathcal{F} : M \rightarrow TM$ is *globally Hamiltonian* if there exists a *Hamiltonian function* $H : M \rightarrow \mathbb{R}$ such that

$$i(\mathcal{F})\omega = dH.$$

We stress two key properties of the symplectic case. First, the generator of a family of exact symplectic maps is globally Hamiltonian and there exists a simple relation between this Hamiltonian and the generating function of the maps. Second, the stable and the unstable invariant manifolds of a connected normally hyperbolic invariant submanifold A of an (exact) symplectic map are (exact) isotropic if and only if A is a fixed point, in which case they are Lagrangian (that is, n -dimensional). The second property is an obstruction to develop a symplectic version of our canonical Melnikov theory for general normally hyperbolic invariant submanifolds; we shall do it just for fixed points.

These properties are well known, but we prove both for completeness.

Proposition 9. *Let f_ϵ be a family of exact symplectic maps with generating function S_ϵ and generating vector field \mathcal{F}_ϵ . Then, \mathcal{F}_ϵ is globally Hamiltonian with Hamiltonian*

$$H_\epsilon = \lambda(\mathcal{F}_\epsilon) - \frac{\partial S_\epsilon}{\partial \epsilon} \circ f_\epsilon^{-1}. \tag{25}$$

Proof. By definition, $\frac{\partial}{\partial \epsilon} f_\epsilon = \mathcal{F}_\epsilon \circ f_\epsilon$, so the Lie derivative with respect to \mathcal{F}_ϵ is $L_{\mathcal{F}_\epsilon} \lambda = (f_\epsilon^*)^{-1} \frac{\partial}{\partial \epsilon} (f_\epsilon^* \lambda)$. Using Cartan’s formula: $L_{\mathcal{X}} \lambda = i(\mathcal{X})d\lambda + di(\mathcal{X})\lambda$, and taking the derivative with respect to ϵ of the relation $dS_\epsilon = f_\epsilon^* \lambda - \lambda$, we get

$$d \frac{\partial S_\epsilon}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} f_\epsilon^* \lambda = f_\epsilon^* L_{\mathcal{F}_\epsilon} \lambda = f_\epsilon^* (i(\mathcal{F}_\epsilon)d\lambda + di(\mathcal{F}_\epsilon)\lambda).$$

Rearranging this yields

$$i(\mathcal{F}_\epsilon)d\lambda = (f_\epsilon^{-1})^* d \frac{\partial S_\epsilon}{\partial \epsilon} - di(\mathcal{F}_\epsilon)\lambda = d \left(\frac{\partial S_\epsilon}{\partial \epsilon} \circ f_\epsilon^{-1} - i(\mathcal{F}_\epsilon)\lambda \right).$$

Finally, $i(\mathcal{F}_\epsilon)\lambda = \lambda(\mathcal{F}_\epsilon)$, since λ is a one-form. □

Proposition 10. *Let $f : M \rightarrow M$ be a diffeomorphism with a connected, normally hyperbolic, invariant manifold A . If f is (exact) symplectic, the stable and unstable invariant manifolds $W^{s,u} = W^{s,u}(A, f)$ are (exact) isotropic if and only if A is a hyperbolic fixed point. Moreover, in this case the submanifolds $W^{s,u}$ are (exact) Lagrangian.*

Proof. Since A is normally hyperbolic, (5) implies that for each $a \in A$, $s = \dim E_a^s$, $u = \dim E_a^u$ and $c = \dim T_a A$ sum to $\dim T_a M = \dim M = 2n$. Since f is symplectic $s = u = n - c/2$. Therefore $\dim W^s = \dim W^u = u + c = n + c/2$. Suppose that $W^{s,u}$ are isotopic; then since isotopic manifolds have maximum dimension n , we must have $c = 0$. Consequently, $\dim A = 0$ and since A is connected it is a hyperbolic fixed point.

Conversely, assume that $\dim A = 0$, so that $A = \{a\}$ for some hyperbolic fixed point a and $\dim W^{s,u} = n$. To prove that W^s is Lagrangian, take any two vectors $u, v \in T_\xi W^s$. We know that $Df^k u$ and $Df^k v$ tend to zero as $k \rightarrow +\infty$, since the stable directions are uniformly contracted. Since f preserves ω , we have

$$\omega(u, v) = (f^k)^* \omega(u, v) = \omega(Df^k u, Df^k v) \rightarrow \omega(0, 0) = 0$$

as $k \rightarrow +\infty$.

If, in addition, f is exact symplectic, then for every closed loop γ , $\oint_\gamma \lambda = \oint_{f(\gamma)} \lambda$. Now suppose that $\gamma \subset W^s$, so that $f^k(\gamma) \rightarrow \{a\}$ as $k \rightarrow +\infty$. Then

$$\oint_\gamma \lambda = \oint_{f^k(\gamma)} \lambda \rightarrow \oint_a \lambda = 0.$$

Finally, the (exact) Lagrangian character of W^u follows from the fact that it is the stable manifold for f^{-1} . □

4.2. Melnikov method for hyperbolic fixed points of exact symplectic maps

Let $f_\epsilon : M \rightarrow M$ be a family of exact symplectic maps such that f_0 has an exact Lagrangian saddle connection Λ between two hyperbolic fixed points a and b . Note that in this case the stable and unstable manifolds are as smooth as the map f_ϵ . We assume that $H_0(a) = H_0(b)$, where H_ϵ is the Hamiltonian (25). Without loss of generality, we can assume that $H_0(a) = H_0(b) = 0$.

The natural measure of splitting for this case is a real valued function L , the *Melnikov potential*, whose derivative measures the splitting [9]. In other words, non-degenerate critical points of the Melnikov potential predict transverse splitting. In this subsection, we shall find the relation between the one-form dL introduced by [38] and the Melnikov displacement.

Definition 8 (Melnikov potential). For a saddle connection Λ and Melnikov displacement $\mathcal{D} : \Lambda \rightarrow \nu(\Lambda)$, the function $L : \Lambda \rightarrow \mathbb{R}$ implicitly defined by

$$dL = j_\Lambda^*(i(\mathcal{D})\omega). \tag{26}$$

is the Melnikov potential.⁴

We will show next that L is indeed defined by (26) and that its critical points correspond to zeros of the displacement.

Proposition 11. The pullback of the one-form $i(\mathcal{D})\omega$ to the saddle connection Λ is well defined and exact. In particular, there exists a function $L : \Lambda \rightarrow \mathbb{R}$, determined uniquely up to additive constants that obeys (26).

Moreover, the set of simple zeros of the Melnikov displacement \mathcal{D} coincides with the set of non-degenerate critical points of the function L .

Remark 5. Let N be a submanifold of M and $\mathcal{X} : N \rightarrow \nu(N)$ a section of its normal bundle. Then the pullback of the one-form $i(\mathcal{X})\omega$ to the submanifold N is well defined if and only if N is isotropic. This has to do with the fact that $j_N^*(i(\mathcal{X})\omega)$ is well defined if and only if

$$\omega(\mathcal{X} + \mathcal{Y}, \mathcal{Z}) = \omega(\mathcal{X}, \mathcal{Z})$$

for any vector fields $\mathcal{X} : N \rightarrow T_N M$ and $\mathcal{Y}, \mathcal{Z} : N \rightarrow T N$.

Proof. The pullback is well defined because Λ is Lagrangian, see remark 5. With regard to the exactness, it suffices to prove that $\oint_\gamma j_\Lambda^*(i(\mathcal{D})\omega) = 0$ for any closed path $\gamma \subset \Lambda$. Let γ be a closed path contained in the saddle connection, then

$$\oint_\gamma j_\Lambda^*(i(\mathcal{D})\omega) = \oint_\gamma j_\Lambda^*(i(f_0^*\mathcal{D})(f_0^*\omega)) = \oint_{f_0(\gamma)} j_\Lambda^*(i(\mathcal{D})\omega),$$

where, since f_0 is symplectic, the Melnikov displacement is invariant under the pullback of f_0 , and the saddle connection is invariant under f_0 . Finally, since $\gamma \in W_0^s(b, f_0)$, we obtain that

$$\oint_\gamma j_\Lambda^*(i(\mathcal{D})\omega) = \oint_{f_0^k(\gamma)} j_\Lambda^*(i(\mathcal{D})\omega) \longrightarrow \oint_b j_\Lambda^*(i(\mathcal{D})\omega) = 0$$

as $k \rightarrow +\infty$.

The equivalence between simple zeros of \mathcal{D} and non-degenerate critical points of L follows from the Lagrangian character of the saddle connection Λ . □

An explicit series for L can be obtained using the Hamiltonian (25).

⁴ For simplicity, henceforth we will write this relation as $i(\mathcal{D})\omega = dL$.

Corollary 12. *The Melnikov potential (26) is given by the absolutely convergent series*

$$L = \sum_{k \in \mathbb{Z}} H_0 \circ f_0^k. \quad (27)$$

Proof. Since f_0 is symplectic, and the generator \mathcal{F}_0 is globally Hamiltonian with Hamiltonian H_0 : $i(\mathcal{F}_0)\omega = dH_0$, we deduce that

$$i(\mathcal{D})\omega = \sum_{k \in \mathbb{Z}} i((f_0^*)^k \mathcal{F}_0)\omega = \sum_{k \in \mathbb{Z}} (f_0^*)^k i(\mathcal{F}_0)\omega = \sum_{k \in \mathbb{Z}} d(H_0 \circ f_0^k) = dL.$$

The series converges absolutely because $H_0(a) = H_0(b) = 0$, $\lim_{k \rightarrow -\infty} f_0^k(\xi) = a$, and $\lim_{k \rightarrow +\infty} f_0^k(\xi) = b$. In fact, it converges at a geometric rate. \square

While the Melnikov potential has been used many times for exact symplectic, twist and Hamiltonian maps, the formulation given here, using the power of deformation theory, is more elegant.

The Melnikov potential introduced here is identical to the one defined in [9]. This can be checked by direct comparison of formula (27) with formula (2.7) of the cited paper, using (25) to express H_0 in terms of the derivative of the generating function S_ϵ at $\epsilon = 0$.

We also note that it is impossible to define a ‘Melnikov potential’ on the saddle connection of normally hyperbolic invariant manifolds with non-zero dimension, because then the saddle connection is not isotropic (proposition 10) and so the identity (26) makes no sense (remark 5). Nevertheless, even in this case the Melnikov displacement is defined.

4.3. Area-preserving maps

In this subsection, we restrict to the two-dimensional case in order to show in a simple way that the Melnikov potential and the classical Melnikov function are transparently related. Moreover, we will see that in some cases there exist geometric obstructions to the non-vanishing of Melnikov functions. These points are most easily seen by choosing a special time-like parametrization of the saddle connection.

We consider the standard symplectic structure on the plane, $(M, \omega) = (\mathbb{R}^2, dx \wedge dy)$, and let J be the standard 2×2 symplectic matrix: $\omega(u, v) = u^T J v = \langle u, Jv \rangle$.

Suppose that $f_\epsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a family of diffeomorphisms preserving area and orientation, and H_ϵ is the generating Hamiltonian for \mathcal{F}_ϵ , the generator for f_ϵ . We assume that the unperturbed map has a saddle connection $\Sigma \subset W^u(a) \cap W^s(b)$ between two hyperbolic fixed points a and b such that $H_0(a) = H_0(b) = 0$. Note that the unperturbed map need not be integrable.

The key point is that, on a one-dimensional saddle connection, there is a parametrization $\alpha : \mathbb{R} \rightarrow \Sigma$ such that

$$f_0(\alpha(t)) = \alpha(t + 1), \quad \lim_{t \rightarrow -\infty} \alpha(t) = a, \quad \lim_{t \rightarrow +\infty} \alpha(t) = b.$$

Remark 6. In many cases, such parametrizations can be expressed in terms of elementary functions, and the Melnikov function can be explicitly computed [8, 14, 23].

Consequently, α provides a diffeomorphism between the saddle connection Σ and the real line, so that objects defined over Σ can be considered as depending on the real variable t . Thus, for example, the Melnikov potential (27) can be replaced by $L \circ \alpha$ to become a function $L : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$L(t) = \sum_{k \in \mathbb{Z}} H_0(\alpha(t + k)). \quad (28)$$

Here we abuse the notation by not giving the function a new name.

Our goal is to show that the classical Melnikov function (24), or rather the composition $M \circ \alpha$, can be computed by differentiating L :

$$M(t) = L'(t). \quad (29)$$

In addition we will show that these functions have the following properties.

- Periodicity: $L(t + 1) = L(t)$ and $M(t + 1) = M(t)$.
- In each fundamental domain $[t, t + 1)$, M must vanish. Indeed, $\int_t^{t+1} M(s) ds = 0$ for any $t \in \mathbb{R}$. (This property will be generalized to volume-preserving maps in the next section.)
- Near each simple zero of $M(t)$ or non-degenerate critical point of $L(t)$ there is transverse intersection of the stable and unstable manifolds.

Given (28) and (29), the first two properties are obvious. Periodicity is simply a consequence of the invariance of (28) under $t \rightarrow t + 1$; this invariance under the unperturbed map is a property of all of the Melnikov objects introduced so far. The second property is a simple consequence of periodicity and integration of (29).

To show that $M(t)$ is actually the classical Melnikov function (24) we must construct an adapted normal vector field η on Λ , recall definition 7. We claim that, when thought of as a function of t , such a vector field $\eta : \mathbb{R} \rightarrow \mathbb{R}^2$ is given by

$$\eta(t) = J\alpha'(t).$$

This claim is proved in lemma 13 at the end of this subsection.

The relation between M and the classical Melnikov function (24) follows from a straightforward computation of the derivative using $J^2 = -I$:

$$\begin{aligned} L'(t) &= \sum_{k \in \mathbb{Z}} \langle \nabla H_0(\alpha(t+k)), \alpha'(t+k) \rangle \\ &= \sum_{k \in \mathbb{Z}} \langle J\alpha'(t+k), -J\nabla H_0(\alpha(t+k)) \rangle \\ &= \sum_{k \in \mathbb{Z}} \langle \eta(t+k), \mathcal{F}_0(\alpha(t+k)) \rangle. \end{aligned}$$

Thus

$$M(t) = \sum_{k \in \mathbb{Z}} \langle \eta(t+k), \mathcal{F}_0(\alpha(t+k)) \rangle,$$

which is the obvious form of (24) under the parametrization α . This verifies (29) and the final property.

To end this subsection, it remains to prove the claim about the vector field η .

Lemma 13. *The vector field $\eta : \Sigma \rightarrow \mathbb{R}^2$ defined by $\eta(\alpha(t)) = J\alpha'(t)$ is an adapted normal vector field on the saddle connection Σ .*

Proof. Since f_0 is symplectic, $Df_0^T J Df_0 = J$. Consequently,

$$\begin{aligned} J\alpha'(t+1) &= Df_0^{-1}(\alpha(t+1))^T J Df_0^{-1}(\alpha(t+1))\alpha'(t+1) \\ &= Df_0^{-1}(\alpha(t+1))^T J\alpha'(t). \end{aligned}$$

Thus, given any vector field $\mathcal{Y} : \Sigma \rightarrow \mathbb{R}^2$, we have

$$\begin{aligned} f_0^* \langle \eta, \mathcal{Y} \rangle (\alpha(t)) &= \langle \eta(\alpha(t+1)), \mathcal{Y}(\alpha(t+1)) \rangle \\ &= \langle J\alpha'(t+1), \mathcal{Y}(\alpha(t+1)) \rangle \\ &= \langle J\alpha'(t), Df_0^{-1}(\alpha(t+1))\mathcal{Y}(\alpha(t+1)) \rangle \\ &= \langle J\alpha'(t), Df_0^{-1}(f_0(\alpha(t)))\mathcal{Y}(f_0(\alpha(t))) \rangle \\ &= \langle \eta, f_0^* \mathcal{Y} \rangle (\alpha(t)). \end{aligned}$$

Moreover, since J is antisymmetric, $\langle \eta, \alpha' \rangle = 0$. Therefore, according to definition 7, $\eta : \Sigma \rightarrow \mathbb{R}^2$ is an adapted normal vector field. \square

5. Volume-preserving maps

For the case of a volume-preserving mapping with a codimension-one saddle connection, an adapted normal field formulation of the Melnikov function also applies [25]. Here we show how to relate this to the Melnikov displacement (7).

Let $f_\epsilon : M \rightarrow M$ be a family of volume-preserving diffeomorphisms on an oriented n -dimensional manifold M with volume form Ω such that f_0 has a codimension-one saddle connection Σ between two normally hyperbolic invariant sets A and B .

We start with a simple lemma about the generator for f_ϵ .

Proposition 14. *Let \mathcal{F}_ϵ be the generator of a volume-preserving smooth family f_ϵ . Then*

- (i) *the divergence of \mathcal{F}_ϵ with respect to Ω is zero,*
- (ii) *the one-form $i(\mathcal{F}_\epsilon)\Omega$ is closed,*
- (iii) *if M is simply connected, then $i(\mathcal{F}_\epsilon)\Omega$ is exact.*

Proof. From [1, theorem 2.2.21], we have that $L_{\mathcal{F}_\epsilon}\Omega = 0$. Since the divergence is defined by $\text{div}(\mathcal{F}_\epsilon)\Omega = L_{\mathcal{F}_\epsilon}\Omega$, this implies that the divergence vanishes. Moreover, since $d\Omega = 0$ and $L_{\mathcal{F}_\epsilon}\Omega = di(\mathcal{F}_\epsilon)\Omega + i(\mathcal{F}_\epsilon)d\Omega$, then $di(\mathcal{F}_\epsilon)\Omega = 0$, implying (ii) and (iii). \square

In order to find an invariant non-degenerate $(n-1)$ -form from any adapted normal field, we assume that M has a Riemannian metric $\langle \cdot, \cdot \rangle$.

Proposition 15. *If η is an adapted vector field (cf definition 7), then*

$$\omega_\eta = \frac{i(\eta)\Omega}{\langle \eta, \eta \rangle} \quad (30)$$

is a non-degenerate $(n-1)$ -form on Σ that is invariant under the restriction $f_0|_\Sigma$.

Proof. By definition η is non-zero, so that ω_η is non-degenerate. To prove that $f_0^*\omega_\eta = \omega_\eta$ on the saddle connection Σ , we introduce the vector field $\mathcal{Z}_\eta : \Sigma \rightarrow T\Sigma$ defined by

$$\mathcal{Z}_\eta \equiv \frac{f_0^*\eta}{\langle \eta, f_0^*\eta \rangle} - \frac{\eta}{\langle \eta, \eta \rangle}.$$

This vector field is tangent to Σ , because $\langle \eta, \mathcal{Z}_\eta \rangle \equiv 0$. Now, we compute the difference

$$f_0^*\omega_\eta - \omega_\eta = \frac{i(f_0^*\eta)f_0^*\Omega}{f_0^*\langle \eta, \eta \rangle} - \frac{i(\eta)\Omega}{\langle \eta, \eta \rangle} = \frac{i(f_0^*\eta)\Omega}{\langle \eta, f_0^*\eta \rangle} - \frac{i(\eta)\Omega}{\langle \eta, \eta \rangle} = i(\mathcal{Z}_\eta)\Omega.$$

Hence, it suffices to see that the $(n-1)$ -form $i(\mathcal{Z}_\eta)\Omega$ vanishes identically on the tangent space $T\Sigma$. This follows from the fact that \mathcal{Z}_η is tangent to Σ and $\dim \Sigma = n-1$. \square

The Melnikov function associated with η is defined using ω_η .

Definition 9 (Volume-preserving Melnikov function). Let \mathcal{D} be the Melnikov displacement (7). Given an adapted vector field η on Σ , we define the Melnikov function $M_\eta : \Sigma \rightarrow \mathbb{R}$ as the unique C^{r-1} function such that

$$M_\eta \omega_\eta = i(\mathcal{D})\Omega \tag{31}$$

as $(n - 1)$ -forms on Σ .

The previous definition first appeared in [25]. Note that $i(\mathcal{D})\Omega$ is an $(n - 1)$ -form, but it is possible that it might be degenerate; if this were the case then zeros of M_η need not correspond to those of \mathcal{D} . We will show next that this is not the case.

Proposition 16. *The Melnikov function M_η is invariant under the map f_0 and $M_\eta = \langle \eta, \mathcal{D} \rangle$. Moreover, a point ξ_0 is a zero of M_η if and only if it is a zero of \mathcal{D} .*

Proof. Using (30), we obtain

$$i(\mathcal{D})\Omega - \langle \eta, \mathcal{D} \rangle \omega_\eta = i(v)\Omega,$$

where $v \equiv \mathcal{D} - c\langle \eta, \mathcal{D} \rangle / \langle \eta, \eta \rangle \eta$. Since $v \in T\Sigma$, we conclude that $i(v)\Omega \equiv 0$, as an $(n - 1)$ -form on Σ , and thus $M_\eta = \langle \eta, \mathcal{D} \rangle$. Since η is an adapted field, and \mathcal{D} is invariant, $f_0^* \langle \eta, \mathcal{D} \rangle = \langle \eta, f_0^* \mathcal{D} \rangle = \langle \eta, \mathcal{D} \rangle$. Therefore, $f_0^* M_\eta = M_\eta$. Finally, $M_\eta(\xi_0) = 0$ only when $\mathcal{D}(\xi_0) = 0$, since η is non-zero and normal to $T\Sigma$, and \mathcal{D} is not in the tangent space. \square

We would like to show, as we did for the area-preserving case in section 4.3, that the volume-preserving Melnikov function necessarily has zeros on Σ . To do this, we will show that the integral of M_η with respect to the measure ω_η is zero. This can be accomplished by dividing the saddle connection into pieces—fundamental domains—that are mapped into each other by f_0 .

Definition 10 (Proper boundary). Let A be a compact normally hyperbolic invariant manifold of a diffeomorphism f_0 with stable manifold $W^s(A)$. A proper boundary, γ , is a submanifold of $W^s(A)$ that bounds an isolating neighbourhood of A in $W^s(A)$. In other words, γ is proper if there is a closed submanifold $W_\gamma^s(A)$ such that

- (i) $\partial W_\gamma^s(A) = \gamma$ and
- (ii) $f_0(W_\gamma^s(A)) \subset \text{int}(W_\gamma^s(A))$.

We refer to the closed set $W_\gamma^s(A)$ as the stable manifold starting at γ .

Similarly, for the unstable manifold, a submanifold $\sigma \subset W^u(A)$ is proper for f_0^{-1} . However, in this case we define the unstable manifold up to σ , denoted by $W_\sigma^u(A)$, as the interior of the local manifold that corresponds to f_0^{-1} .

Notice that the definition is not symmetric, because $W_\gamma^s(A)$ is a closed subset of $W^s(A)$, while $W_\sigma^u(A)$ is open in $W^u(A)$. The asymmetry is just a technicality in order to simplify some proofs.

Definition 11 (Fundamental domain). Let A be a hyperbolic invariant set for f_0 . A submanifold with boundary P is a fundamental domain of $W^s(A)$ if there exists some proper boundary $\gamma \in W^s(A)$ such that

$$P = W_\gamma^s(A) \setminus W_{f_0 \circ \gamma}^s(A).$$

Equivalently, a fundamental domain in $W^u(A)$ is a manifold with boundary of the form

$$P = W_\sigma^u(A) \setminus W_{f_0^{-1} \circ \sigma}^u(A),$$

where σ is a proper boundary in $W^u(A)$.

In each case, the fundamental domain is a manifold with boundary $\partial\mathcal{P} = \gamma \cup f_0(\gamma)$. An immediate consequence of the definition is that all the forward and backward iterations of a fundamental domain are also fundamental. It is easy to see that proper boundaries always exist, and in fact, the unstable manifold can be decomposed as the disjoint union of fundamental domains:

$$W^s(A) \setminus A = \bigcup_{k \in \mathbb{Z}} f_0^k(P).$$

The importance of fundamental domains is that much of the information about the entire manifold can be found by looking only at these submanifolds. For example, as discussed in [27], the topology of the intersections of W^u and W^s can be studied by restricting to P .

Proposition 17. *Let M be a simply connected manifold and $f_\epsilon : M \rightarrow M$ a family of volume-preserving maps such that f_0 has a saddle connection Σ with fundamental domain P . If ω_η is the $(n - 1)$ -form defined in (30) and M_η is the Melnikov function (31), then $\int_P M_\eta \omega_\eta = 0$.*

Proof. The fundamental domain P is a submanifold with boundary, such that $\partial P = \gamma \cup f_0(\gamma)$, where γ is a closed curve that does not intersect $f_0(\gamma)$. If we give an orientation $[P]$ to P , the induced orientation on the boundary satisfies $[\gamma] = -[f_0(\gamma)]$. According to proposition 14, the form $i(\mathcal{F}_0)\Omega$ is exact. Thus there exists an $(n - 2)$ -form Ξ such that $d\Xi = i(\mathcal{F}_0)\Omega$. From (8) we conclude that

$$i(\mathcal{D})\Omega = i\left(\sum_{k \in \mathbb{Z}} (f_0^*)^k \mathcal{F}_0\right)\Omega = \sum_{k \in \mathbb{Z}} (f_0^*)^k (i(\mathcal{F}_0)\Omega) = d \sum_{k \in \mathbb{Z}} (f_0^*)^k \Xi = d\Psi,$$

where $\Psi = \sum_{k \in \mathbb{Z}} (f_0^*)^k \Xi$. Finally, from definition 9, Stokes's theorem and the invariance $f_0^*\Psi = \Psi$, we get that the integral

$$\int_P M_\eta \omega_\eta = \int_P i(\mathcal{D})\Omega = \int_P d\Psi = \int_{\partial P} \Psi = \int_\gamma \Psi - \int_\gamma f_0^*\Psi$$

vanishes. \square

The previous result implies that the stable and unstable manifolds of a perturbed saddle connection necessarily intersect. Examples of such intersections were computed for the case $M = \mathbb{R}^3$ —where the hypothesis of proposition 17 are satisfied—in [25, 27].

6. Conclusion and future research

We have studied a general theory of the Melnikov method that can be applied to many different settings. Formula (8) for the Melnikov displacement \mathcal{D} generalizes many of the classical methods that use a normal vector field to measure the displacement with respect to a natural direction. For example, when the saddle connection is defined as the level set of a first integral I , the classical Poincaré function $M_I = dI(\mathcal{D})$ measures the splitting as the rate of change of the first integral. If there is a Riemannian structure and an associated adapted normal vector field η , then the classical Melnikov function $M_\eta = \langle \eta, \mathcal{D} \rangle$ measures the rate of change of the splitting in this normal direction. For the case of exact symplectic maps with saddle connections between hyperbolic fixed points, the Melnikov potential L is defined on a Lagrangian submanifold and acts as a generator for the displacement: $dL = i(\mathcal{D})\omega$.

This Melnikov theory can be extended to other situations and can be applied in many problems. For instance, one can study billiard dynamics inside a perturbed ellipsoid, following a program initiated in [3, 6]. It turns out that the billiard map inside an ellipsoid is an

exact symplectic diffeomorphism defined on the cotangent bundle of the ellipsoid, which has a two-dimensional normally hyperbolic invariant manifold with a three-dimensional saddle connection. Therefore, ellipsoidal billiards represent a strong motivation for a more detailed study of general normally hyperbolic invariant manifolds in a symplectic framework. This is a work in progress.

In discrete volume geometry there are many open Melnikov problems, since the application of Melnikov methods to volume-preserving maps began just a few years ago [25, 27]. We plan to continue this program in several ways. As a first step, we plan to obtain bounds on the number of primary heteroclinic orbits in terms of the degree of the polynomial perturbation [28].

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