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GENERATING FORMS FOR EXACT VOLUME-PRESERVING MAPS

H. E. Lomelí[1]

Department of Mathematics
Instituto Tecnológico Autónomo de México, Mexico, DF 01000

J. D. Meiss

Department of Applied Mathematics
University of Colorado, Boulder, CO 80309-0526, USA

Abstract. We develop a general theory of implicit generating forms for volume-preserving diffeomorphisms on a manifold. Our results generalize the classical formulas for generating functions of symplectic twist maps and examples of Carroll for volume-preserving maps on $\mathbb{R}^n$.

1. Introduction.

A C^1 map $f : M \to M$ preserves a volume form Ω on a manifold M if it satisfies

$$f^*\Omega = \Omega \ .$$

For example, if $M = \mathbb{R}^n$ and the volume form is $\Omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$, then f is volume preserving when its Jacobian has unit determinant, $\det(Df) = 1$.

The study of such maps is interesting on the one hand because volume-preserving maps are a simple and natural higher-dimensional generalization of the much-studied class of area-preserving maps. On the other hand, the infinite dimensional group of volume-preserving diffeomorphisms on $\mathbb{R}^3$ is at the core of the ambitious program to reformulate hydrodynamics [2]. Volume-preserving maps arise in a number of applications such as the study of the motion of Lagrangian tracers in incompressible fluids or of the structure of magnetic field lines [16, 17, 29, 25].

In this paper we will study the construction of *generating forms* for exact volume-preserving maps. A similar construct, *generating functions*, is familiar in the exact symplectic case. Recall that a symplectic map f preserves a nondegenerate, closed two-form $\omega = dq \wedge dp$ defined on an $n = 2d$ dimensional manifold: $f^*\omega = \omega$. Exact symplectic maps arise when ω is exact. For instance when there exists a *Liouville one-form* ν on a cotangent bundle, the symplectic form is defined by $\omega = -d\nu$. Then f is an exact symplectic map if

$$f^*\nu - \nu = dL \ , \tag{1}$$

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[1] *Current address:* Department of Mathematics, The University of Texas, Austin, TX 78712

for a "generating" function L defined on M. If we denote the map by

$$(Q, P) = f(q, p) ,$$

and choose, e.g., $\nu = pdq \equiv \sum_{i=1}^{d} p_i dq_i$, then (1) has the form

$$PdQ - pdq = dL . \tag{2}$$

The theory of canonical generating functions regards this as an equation not on M, but on the doubled phase space $N = M \times M$ with coordinates (q, p, Q, P) [1, 9]. In this case, (2) is not valid everywhere on N, but only on the graph $F = \{(q, p, Q, P) \mid (Q, P) = f(q, p)\} \subset N$ of f. Alternatively, we say that $L : N \to \mathbb{R}$ is a generating function, with respect to ν, if the set on which (1) is satisfied is precisely the graph F. The resulting map f is necessarily exact symplectic.

There are four special cases of (1) that are typically defined [13]. For example, if L is assumed to depend only upon (q, Q), then (2) is equivalent to the implicit equations

$$p = -\partial_q L(q, Q) ,$$
$$P = \partial_Q L(q, Q) .$$

These generate a map when the implicit equations can be solved for (Q, P); this occurs under a *twist condition*, $\det(\partial_{qQ} L(q, Q)) \neq 0$, that is, the matrix of partial derivatives $\partial_{q_i Q_j} L(q, Q)$ is non degenerate, typically either positive or negative definite. Notice that the non degeneracy of the matrix $\partial_{qQ} L(q, Q)$ is a necessary condition, but in many cases not sufficient. For some possible sufficient conditions, see [14].

Many other generating functions can be obtained by other choices of the form ν [3].

In the following sections, we analyze the group of exact volume-preserving diffeomorphisms and obtain implicit generating *forms* for exact volume-preserving maps by mimicking the symplectic construction. In particular, in some cases it is possible to determine an exact volume-preserving diffeomorphism f from an $(n-2)$-form Λ on $N = M \times M$.

We start with a discussion of exact volume-preserving maps in §2. Examples of generating forms were first given—as far as we know—by Carroll [6], see §4, though he did not use the notation of differential forms. The general formulation is given in §5, and additional examples are presented in the following sections. A volume form is not always exact, for example when $M = \mathbb{T}^n$. However, in some cases a generating form can still be obtained on the universal cover of M as we discuss in §8. Applications to maps on $\mathbb{T}^d \times \mathbb{R}^k$ are given in the last section.

2. **Exact volume-preserving maps.**

A volume form Ω is exact when there exists an $(n-1)$-form α such that $\Omega = d\alpha$. For this case, exact volume-preserving maps can be defined by analogy with the symplectic case.

Definition 2.1. *Let (M, Ω) be a manifold on which the volume form Ω is exact and suppose that $d\alpha = \Omega$. A diffeomorphism $f : M \to M$ is $\alpha-$*exact *volume preserving if there exists an $(n-2)$-form λ on M such that*

$$f^*\alpha - \alpha = d\lambda. \tag{3}$$

We will denote by $\mathrm{Diff}_\alpha(M)$ the set of $\alpha-$exact volume-preserving diffeomorphisms.

If α is clear from the context, we will drop α and simply say that f is exact.

It is clear that if f is exact volume preserving, then f^{-1} is also. Moreover, if $f = f_1 \circ f_2$ is the composition of exact volume-preserving maps with forms λ_1 and λ_2, respectively, then since $(f_1 \circ f_2)^* = f_2^* f_1^*$,

$$f^*\alpha - \alpha = f_2^*(f_1^*\alpha - \alpha) + f_2^*\alpha - \alpha = d(f_2^*\lambda_1 + \lambda_2) .$$

Thus f is exact volume-preserving with

$$\lambda = f_2^*\lambda_1 + \lambda_2 . \tag{4}$$

Therefore $\mathrm{Diff}_\alpha(M)$ is a group that can be regarded as an infinite dimensional Lie group.

Clearly, if there are two forms, say α and $\tilde{\alpha}$, such that $\alpha - \tilde{\alpha}$ is exact, then either can be used in (3), and moreover, $\mathrm{Diff}_\alpha(M) = \mathrm{Diff}_{\tilde{\alpha}}(M)$. However, Def. 2.1 can be generalized slightly for the case that $\alpha - \tilde{\alpha}$ is not exact.

Definition 2.2. *Suppose that $d\alpha = d\tilde{\alpha} = \Omega$. A diffeomorphism $f : M \to M$ is exact volume preserving* with respect to $(\alpha, \tilde{\alpha})$ *if*

$$f^*\tilde{\alpha} - \alpha = d\lambda , \tag{5}$$

for some form λ.

Notice that a diffeomorphism is $\alpha-$exact if it is exact with respect to (α, α). When $d\tilde{\alpha} = d\alpha = \Omega$, the difference $\tilde{\alpha} - \alpha$ is closed. If this difference is also exact, we can set $\tilde{\alpha} = \alpha + d\beta$, so that if f is exact volume preserving with respect to the pair $(\alpha, \tilde{\alpha})$, it is also $\alpha-$exact:

$$f^*\alpha - \alpha = f^*(\tilde{\alpha} - d\beta) - \alpha = d(\lambda - f^*\beta) . \tag{6}$$

Therefore when $\tilde{\alpha} - \alpha$ is exact, then being exact with respect to $(\alpha, \tilde{\alpha})$ is equivalent to being $\alpha-$exact. Thus on $\mathbb{R}^n$ Def. 2.2 gives nothing new, since every closed form is exact; however, more generally the difference $\tilde{\alpha} - \alpha$ need not be exact, and the set of diffeomorphisms exact with respect to $(\alpha, \tilde{\alpha})$ is not a group.

As we will see in §4, the generalization (5) gives rise to distinct classes of implicit generating forms even on $\mathbb{R}^n$. On more general manifolds additional care must be taken, see §8.

When f is α-exact, the form λ can be used to compute volumes of invariant or partially invariant regions. For example, suppose that $\mathcal{C}$ is an orientable, boundary-free, codimension-two manifold that is invariant under f, e.g., if $\dim M = 3$ then $\mathcal{C}$ is an invariant circle. Let $\mathcal{S}$ be any codimension-one embedded submanifold bounded by $\mathcal{C}$ and let $\mathcal{R}$ be the "region" bounded by $\mathcal{S}$ and its image, $\partial\mathcal{R} = f(\mathcal{S}) - \mathcal{S}$. In other words, suppose that a region $\mathcal{R}$ is bounded by $\mathcal{S}$ and its image, with the appropriate orientations. The (algebraic) volume of $\mathcal{R}$ is

$$Vol(\mathcal{R}) = \int_{\mathcal{R}} \Omega = \int_{\mathcal{S}} f^*\alpha - \alpha = \int_{\mathcal{C}} \lambda .$$

Generalizations of this formula can also be used to compute the flux of orbits escaping from a resonance zone in terms of the integral of the form λ along heteroclinic intersections of stable and unstable manifolds [21]. Similar formulas have been extensively used in the symplectic case [22, 23, 8] and should prove useful in studies of volume-preserving transport [27, 11, 4].

Any exact symplectic map of a two-dimensional manifold is exact volume-preserving with the volume form $\Omega = \omega$, if we choose the one-form $\alpha = -\nu$ and the zero-form $\lambda = -L$. This also holds more generally.

Lemma 2.3. *Any exact symplectic diffeomorphism is exact volume-preserving.*

Proof. Suppose (M, ω) is a symplectic manifold of dimension $2d$ and $f : M \to M$ is an exact symplectic diffeomorphism. The d-fold wedge of the two-form ω is a volume form[2]

$$\Omega = \omega^{\wedge d} \equiv \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{d}.$$

Defining $\alpha = -\nu \wedge \omega^{\wedge(d-1)}$ then $d\alpha = \Omega$, and

$$\begin{aligned} f^*\alpha - \alpha &= f^*\left(-\nu \wedge \omega^{\wedge(d-1)}\right) + \nu \wedge \omega^{\wedge(d-1)} = -(f^*\nu - \nu) \wedge \omega^{\wedge(d-1)} \\ &= -dL \wedge \omega^{\wedge(d-1)} = d(-L\omega^{\wedge(d-1)}) \,, \end{aligned}$$

since $f^*\omega = \omega$ and $d\omega = 0$. Thus f is exact volume preserving with the $2(d-1)$ form $\lambda = -L\omega^{\wedge(d-1)}$. ☐

3. **Exact incompressible vector fields.** One important aspect of the structure of Lie groups is the study of their one-parameter subgroups. Here we consider the subgroups of $\mathrm{Diff}_\alpha(M)$ generated by exact incompressible vector fields. Recall that an incompressible vector field X satisfies $L_X\Omega \equiv (\nabla \cdot X)\Omega = 0$, where L_X is the Lie derivative. In other words, X is incompressible if and only if the corresponding flow φ_t generated by X is volume-preserving for each t. To find a similar condition for exact volume-preserving flows, suppose that the flow φ_t is an α-exact volume-preserving diffeomorphism for each t. According to Def. 2.1 there exists a C^1 family of $(n-2)$-forms λ_t such that

$$\varphi_t^*\alpha - \alpha = d\lambda_t \,. \tag{7}$$

Differentiating with respect to t gives

$$\begin{aligned} \frac{d}{dt}\varphi_t^*\alpha &= \varphi_t^* L_X \alpha \\ &= \varphi_t^*\left(i_X d\alpha + d i_X \alpha\right) = d\left(\frac{\partial}{\partial t}\lambda_t\right) \,. \end{aligned}$$

Since $d\alpha = \Omega$, this gives $i_X\Omega = d\left(\varphi_{-t}^*\frac{\partial}{\partial t}\lambda_t - i_X\alpha\right)$; consequently, the flow generated by the vector field X is exact volume-preserving if and only if $i_X\Omega$ is exact.

We will argue that the expression $d\left(\varphi_{-t}^*\frac{\partial}{\partial t}\lambda_t\right)$ does not depend on time. Using the group property of the flow φ_t in (7), it follows that for all $t, s \in \mathbb{R}$.

$$d\lambda_{s+t} = d\left(\varphi_s^*\lambda_t + \lambda_s\right) \,. \tag{8}$$

Letting $\beta = \frac{\partial}{\partial t}\big|_{t=0}\lambda_t$, then differentiating (8) with respect to t and setting $t = 0$ gives

$$d\left(\frac{\partial \lambda_s}{\partial s}\right) = d\left(\varphi_s^*\beta\right). \tag{9}$$

Therefore $d\left(\varphi_{-t}^*\frac{\partial}{\partial t}\lambda_t\right) = d\beta$ and $L_X\alpha = d\beta$. Consequently $i_X\Omega = d\left(\beta - i_X\alpha\right)$ is exact. Moreover, (9) shows that the form

$$\lambda_t - \int_0^t \varphi_\tau^*\beta \, d\tau$$

[2] The standard volume would be $\frac{(-1)^{\lfloor d/2 \rfloor}}{d!}\Omega$.

is closed and, without loss of generality, we can choose

$$\lambda_t = \int_0^t \varphi_\tau^* \beta \, d\tau \ . \tag{10}$$

Notice that λ_t is as smooth as β in all the variables that are different from t. We summarize these results as a proposition.

Proposition 3.1. *Let X be a vector field on a smooth manifold M of dimension n with an exact volume form Ω, such that $\Omega = d\alpha$, for some fixed $(n-1)$-form α. Let φ_t be the flow generated by X, and suppose that it is complete. Then the following are equivalent.*

1. *$i_X \Omega$ is exact.*
2. *There exists an $(n-2)$-form β_X such that $L_X \alpha = d\beta_X$.*
3. *For each $t \in \mathbb{R}$, there exists an $(n-2)$-form λ_t, (10), such that (7) is satisfied.*

One consequence is that a vector field X is exact incompressible independently of α. If a vector field on a smooth manifold M satisfies any of the conditions of Prop. 3.1, we will say that X is an *exact incompressible* vector field with structure form β_X. Another term that has been used for these vector fields is "globally Liouville," see for example [12].

These vector fields have a Lie algebraic structure.

Lemma 3.2. *If X, Y, Z are exact incompressible vector fields, with structure forms β_X, β_Y and β_Z then,*

1. *the Lie bracket $[X, Y]$ is an exact incompressible vector field with structure form*
$$\beta_{[X,Y]} = L_X \beta_Y - L_Y \beta_X \tag{11}$$
and,
2. *if f is exact volume-preserving with $f^*\alpha - \alpha = d\lambda$, the pull-back f^*Z is an exact incompressible vector field with structure form*
$$\beta_{f^*Z} = f^*\beta_Z - L_{f^*Z}\lambda \ . \tag{12}$$

An interesting exercise is to check that the formulas (11) and (12) are compatible. Since $f^*[X,Y] = [f^*X, f^*Y]$, it must be the case the form $\beta_{f^*[X,Y]} - \beta_{[f^*X,f^*Y]}$ is closed. In fact it is possible to show that $\beta_{f^*[X,Y]} = \beta_{[f^*X,f^*Y]}$ directly from the lemma.

Example. Consider the case of a nonautonomous Hamiltonian flow, generated by a C^2 function $H : \mathbb{R}^3 \to \mathbb{R}$. With $H(q, p, \theta)$ we form the autonomous Hamiltonian vector field given by

$$X_H = (H_p, -H_q, 1)^T \ .$$

The volume form is $\Omega = dq \wedge dp \wedge d\theta$ and we can choose $\alpha = -pdq \wedge d\theta$ so that $d\alpha = \Omega$. Hence,

$$i_{X_H}\Omega = dH \wedge d\theta + dq \wedge dp \ ,$$
$$i_{X_H}\alpha = -p\frac{\partial H}{\partial p} d\theta + pdq \ .$$

These imply

$$L_{X_H}\alpha = i_{X_H}\Omega + di_{X_H}\alpha = d\left(H - p\frac{\partial H}{\partial p}\right) \wedge d\theta \ .$$

so that we can define $\beta = -\mathcal{L}_H d\theta$ where

$$\mathcal{L}_H = pH_p - H$$

is the Lagrangian, and

$$\lambda_t = -\left(\int_0^t \mathcal{L}_H \circ \varphi_\tau \, d\tau\right) d\theta \ .$$

It is possible to show directly from equation (11) that

$$\beta_{[X_H, X_G]} = \mathcal{L}_{\{H,G\}} d\theta,$$

where $\{G, H\} = G_q H_p - G_p H_q$ is the Poisson bracket.

4. **Generating forms: Carroll's example.** In this section, we consider the simplest case where $M = \mathbb{R}^3$ and $\Omega = dx \wedge dy \wedge dz$. We will assume that the diffeomorphism $(X, Y, Z) = f(x, y, z)$ is volume-preserving, $f^*\Omega = dX \wedge dY \wedge dZ = \Omega$, and in addition, is exact with respect to $(\alpha, \tilde{\alpha})$, recall (5). Here we give a simple example, based on that of Carroll [6], of an implicit generating form for f.

For $\mathbb{R}^3$ there are three natural choices for these two-forms: $xdy \wedge dz$, $ydz \wedge dx$, and $zdx \wedge dy$. Therefore for α and $\tilde{\alpha}$, there are together, nine simple choices. Essentially we are choosing a subset of the variables (x, y, z, X, Y, Z)—some "old" and some "new"—and combining them in a single function. We will select the one-form λ to depend explicitly on the variables chosen.

For example let us choose $\tilde{\alpha} = zdx \wedge dy$ and $\alpha = xdy \wedge dz$ and try to find a diffeomorphism $f(x, y, z) = (X, Y, Z)$ such that

$$f^*\tilde{\alpha} - \alpha = ZdX \wedge dY - xdy \wedge dz = d\lambda \ , \tag{13}$$

assuming that the one-form λ can be written as

$$\lambda = \Phi(y, z, X, Y)dy + \Psi(y, z, X, Y)dY \ . \tag{14}$$

Here λ is to be thought of as a one-form on $\mathbb{R}^3$; that is, it must be evaluated on the transformation $(X, Y, Z) = f(x, y, z)$. However, we ignore that for the moment and treat (y, z, X, Y) as four independent variables. The differential of (14) is

$$d\lambda = \partial_z \Phi dz \wedge dy + \partial_X \Phi dX \wedge dy + (\partial_Y \Phi - \partial_y \Psi) dY \wedge dy + \partial_z \Psi dz \wedge dY + \partial_X \Psi dX \wedge dY \ .$$

To be consistent with (13), Φ must independent of X, Ψ independent of z, and

$$x = \partial_z \Phi(y, z, Y) \ , \quad \partial_Y \Phi(y, z, Y) = \partial_y \Psi(y, X, Y) \ , \quad Z = \partial_X \Psi(y, X, Y) \ . \tag{15}$$

These three equations locally define a map f, provided that the first equation can be inverted for Y, which requires that $\partial_{zY}\Phi \neq 0$, and that the second can be solved for X, which requires $\partial_{yX}\Psi \neq 0$. By analogy with the case of symplectic maps, we call these conditions *twist conditions*. The twist conditions are geometrical properties of f and f^{-1}, namely

$$\frac{\partial Y}{\partial x} \neq 0 \ , \quad \frac{\partial y}{\partial Z} \neq 0 \ .$$

Note that these conditions need not be satisfied for every volume preserving map f; therefore, a generating form of type (14) will not exist—even locally—for every exact volume preserving diffeomorphism.

The map f is globally defined by the generating form λ if for each (y, z) the image of the line $L_{y,z} = \{(s, y, z) \mid s \in \mathbb{R}\}$ intersects every plane $P_Y = \{(u, Y, v) \mid (u, v) \in \mathbb{R}^2\}$ exactly once. Conversely, for each (X, Y) the preimage of the line $L_{X,Y} = \{(X, Y, s) \mid s \in \mathbb{R}\}$ intersects every plane P_y exactly once. This is shown in Fig. 1.

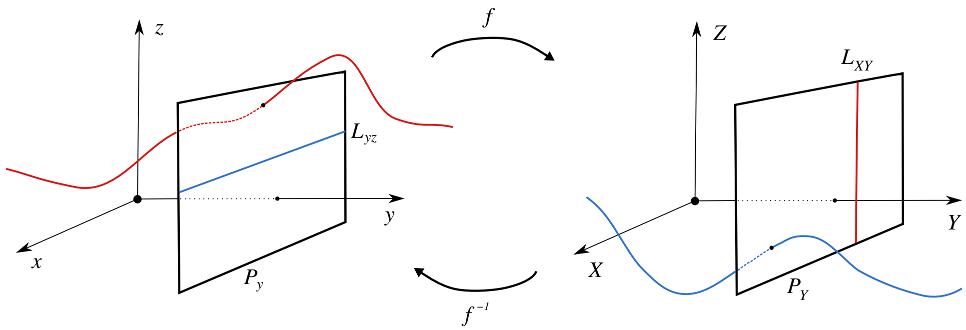

FIGURE 1. Illustration of the twist conditions for the map (15)

As an example, let $\Phi(y,z,Y) = zY + g(y,Y)$ and $\Psi(y,X,Y) = Xy$. Then the map generated by (15) is

$$(X,Y,Z) = (z + \partial_2 g(y,x), x, y) \ ,$$

which is of the form of the shift-like diffeomorphisms studied in [20, 5, 18, 15]. It trivially satisfies the twist conditions since $Y(x,y,z) = x$ and $y(X,Y,Z) = Z$.

5. Generating forms.

Though we thought of λ in §4 as a form on $M = \mathbb{R}^3$, it is more properly thought of as a form on the product space $N = M \times M$. Thus if (x,y,z,X,Y,Z) are the coordinates of a point in N, the expression (14) becomes a one form on N. To distinguish this new form from the original form on M, we will call it Λ.

More generally, let $\tilde{\alpha}$ and α be two $(n-1)$-forms such that $d\alpha = d\tilde{\alpha} = \Omega$. Let $\pi_{1,2} : N \to M$ be the projections

$$\pi_1(m_1,m_2) = m_1 \ , \quad \pi_2(m_1,m_2) = m_2 \ .$$

Following the symplectic case [1, 9], a generating form will be constructed using $\pi_2^*\tilde{\alpha} - \pi_1^*\alpha$, which is an $(n-1)$-form on N. Note that if $(\xi_1,\xi_2) \in T_{(m_1,m_2)}N$, then $(\pi_2^*\tilde{\alpha} - \pi_1^*\alpha)_{(m_1,m_2)}(\xi_1,\xi_2) = \tilde{\alpha}_{m_2}(\xi_2) - \alpha_{m_1}(\xi_1)$.

Definition 5.1 (Generating Form). *An $(n-2)$-form Λ on $N = M \times M$ is a* generating form *with respect to the pair $(\alpha,\tilde{\alpha})$ if the set $F \subset N$ on which the form*

$$\Gamma \equiv \pi_2^*\tilde{\alpha} - \pi_1^*\alpha - d\Lambda \qquad (16)$$

vanishes is the graph $F = \{(m, f(m)) \mid m \in M\}$ of a C^1 function $f : M \to M$. In this case, we will say that the map f is generated *by Λ.*

A similar construction also applies, of course, to symplectic maps as well [30]. Indeed, the idea of using a form to define a submanifold is very old, going back to the question of solving Pfaffian equations to define subbundles of a vector bundle, in particular of the tangent bundle [19]. Our situation does not correspond to the Pfaffian, since we are dealing with the zero-set of a form considered as a section. A Pfaffian usually has constant rank, so its zero-set would be empty.

The notion of Def. 5.1 is equivalent to that of (5). Indeed, if $j : M \to N$ represents the embedding $j(m) = (m, f(m))$, note that $\pi_1 \circ j = id_M$ and $\pi_2 \circ j = f$. The implication is that a generated map is exact.

Proposition 5.2. *If Λ generates a map f with respect to $(\alpha, \tilde{\alpha})$ and $d\alpha = d\tilde{\alpha} = \Omega$, then f is exact volume-preserving with respect to $(\alpha, \tilde{\alpha})$.*

Proof. If we let $\lambda = j^*\Lambda$, then $f^*\tilde{\alpha} - \alpha = d\lambda$. □

Moreover, when f is invertible there is a simple relation between the generating function of a map f and its inverse.

Proposition 5.3. *If f is invertible and is exact volume preserving with respect to $(\alpha, \tilde{\alpha})$ with generator Λ_f, then f^{-1} is exact with respect to $(\tilde{\alpha}, \alpha)$ with generator*

$$\Lambda_{f^{-1}} = -\sigma^* \Lambda_f \ ,$$

where $\sigma : N \to N$ is the permutation $\sigma(m_1, m_2) = (m_2, m_1)$.

When $\alpha - \tilde{\alpha}$ is exact, the exactness with respect to $(\alpha, \tilde{\alpha})$ implies exactness with respect to (α, α), recall (6). This property gives also holds for generating forms.

Lemma 5.4 (Legendre Transformations)**.** *If Λ is a generating form for f with respect to $(\alpha, \tilde{\alpha})$ then $\Lambda + \pi_2^*\tilde{\beta} - \pi_1^*\beta$ is as well, with respect to the pair $(\alpha + d\beta, \tilde{\alpha} + d\tilde{\beta})$ for any $(n-2)$-forms β and $\tilde{\beta}$.*

Proof. It is enough to notice from (16) that

$$\Gamma = \pi_2^* \left(\tilde{\alpha} + d\tilde{\beta}\right) - \pi_1^* \left(\alpha + d\beta\right) - d\left(\Lambda + \pi_2^*\tilde{\beta} - \pi_1^*\beta\right) . \tag{17}$$

□

This property is analogous to the Legendre transformations between various symplectic generating functions [3, 13]. For example in $\mathbb{R}^n$, any *even* permutation,

$$p_{(i)}(x_1, x_2, \ldots, x_n) = (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \ ,$$

is an exact volume-preserving diffeomorphism. Thus, if $\alpha_{(i)} = p_{(i)}^*\alpha$, then there is an $(n-2)$-form $\beta_{(i)}$ such that $\alpha_{(i)} - \alpha = d\beta_{(i)}$. Consequently, if Λ generates f with respect to (α, α), then (17) gives new generators with the permuted forms

$$\pi_2^*\alpha_{(i)} - \pi_1^*\alpha_{(j)} = d\Lambda_{(i),(j)}$$

where

$$\Lambda_{(i),(j)} = \Lambda + \pi_2^*\beta_{(i)} - \pi_1^*\beta_{(j)} \ . \tag{18}$$

In this way, beginning with a basic form, say $\alpha = x_1 dx_2 \wedge \ldots \wedge dx_n$, and an associated generator, we can obtain generators for the $\frac{1}{2}n!$ evenly permuted forms $\alpha_{(i)} = x_{i_1} dx_{i_2} \wedge \ldots \wedge x_{i_n}$. Since each permutation can be done on each copy of M, there are $(\frac{1}{2}n!)^2$ possible choices for $(\alpha, \tilde{\alpha})$ in (16).

6. Thirty-six generating forms on $\mathbb{R}^3$.

For $\mathbb{R}^3$, we will begin with the basic form $zdx \wedge dy$, and by even permutation construct the two additional forms $xdy \wedge dz$, and $ydz \wedge dx$. Since any of the three can be used as well for $\tilde{\alpha}$, there are nine choices for the form Γ. For each such choice, we will see that there are four possible representations for Λ. Thus overall we will find thirty-six different generating forms. These are analogous to the four basic generating functions for area-preserving maps [3, 13].

To catalog the possibilities, begin by choosing $\tilde{\alpha} = \alpha = zdx \wedge dy$, and consider the generating equation

$$\Gamma = ZdX \wedge dY - zdx \wedge dy - d\Lambda = 0 \tag{19}$$

on the graph of a C^1 function $F = \{(m, f(m)) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid m \in \mathbb{R}^3\}$. A general one-form on N will have terms for each of the six coordinates (x, y, z, X, Y, Z); however, to be consistent with (19), $d\Lambda$ can have no terms involving dz and dZ. This implies that any z and Z dependence of Λ can be collected into terms that are total differentials: these give no contribution to the determination of f. Consequently, we set $\Lambda = Adx + Bdy + CdX + DdY$ where the functions A, B, C, and D depend only upon the four variables (x, y, X, Y). Substitution into (19) then gives six equations. The first two are dynamical in nature,

$$z = \partial_y A - \partial_x B \ , \qquad\qquad Z = \partial_X D - \partial_Y C \ ,$$

and the last four are the implicit consistency equations

$$\partial_X A = \partial_x C \ , \qquad\qquad \partial_Y A = \partial_x D \ ,$$
$$\partial_X B = \partial_y C \ , \qquad\qquad \partial_Y B = \partial_y D \ .$$

There is a redundancy in these consistency equations that can be traced to the definition of Λ. Indeed, if we were to impose any one of these equations from the outset, we could rewrite Λ as a form containing only two terms, up to a perfect differential. For example, the first consistency equation implies that $Adx + CdX = d\zeta - \partial_y \zeta dy - \partial_Y \zeta dY$ where—since $\partial_X A = \partial_x C$—we can set $\zeta = \int CdX = \int Adx$. Thus $\Lambda = d\zeta + \bar{B}dy + \bar{D}dY$ with $\bar{B} = B - \partial_y \zeta$ and $\bar{D} = D - \partial_Y \zeta$. Since $d\zeta$ will have no effect on the generating equation (19), this term can be ignored; consequently upon dropping the "bars", Λ becomes $Bdy + DdY$. Since A and C have effectively been set to zero, two of the remaining consistency equations now reduce to $\partial_X B = \partial_x D = 0$, which imply that $\Lambda = B(x, y, Y)dy + D(y, X, Y)dY$. There remain three equations to implicitly determine the three components of the map $(X, Y, Z) = f(x, y, z)$:

$$z = -\partial_x B(x, y, Y) \ , \quad \partial_Y B(x, y, Y) = \partial_y D(y, X, Y) \ , \quad Z = \partial_X D(y, X, Y) \ .$$

This map is well-defined only if these implicit equations can be inverted. The first equation can be solved for $Y(x, y, z)$ only if $\partial_{xY} B \neq 0$, and the second can then be solved for $X(x, y, z)$ only if $\partial_{yX} D \neq 0$. More specifically f must satisfy two conditions: the curves $C = \{Y(x, y, z) \mid z \in \mathbb{R}\}$, and $\tilde{C} = \{y(X, Y, Z) \mid Z \in \mathbb{R}\}$ must be bijections onto $\mathbb{R}$ for each fixed (x, y) and (X, Y), respectively. This will occur, for example, if the derivatives $\partial Y/\partial z$ and $\partial y/\partial Z$ are uniformly positive and bounded:

$$0 < \ell_1 \leq \frac{\partial Y}{\partial z} \ , \ \frac{\partial y}{\partial Z} \leq \ell_2 < \infty \ .$$

A similar reduction of Λ to two terms can be performed by imposing each of the remaining three consistency equations, giving four basic generating forms as shown in Tbl. 1. These four are geometrically distinct in that they have distinct twist conditions.

Additional generating forms can be obtained from Tbl. 1 using the Legendre transformation (18) to change the forms $\pi_2^* \alpha = ZdX \wedge dY$ and $\pi_1^* \alpha = zdx \wedge dy$ into the eight remaining permutations. Specifically let $p_{(231)}(x, y, z) = (y, z, x)$ and $p_{(312)}(x, y, z) = (z, x, y)$ denote the even permutations. Then

$$p_{(231)}^* \alpha - \alpha = xdy \wedge dz - zdx \wedge dy = d(-xzdy) \ ,$$
$$p_{(312)}^* \alpha - \alpha = ydz \wedge dx - zdx \wedge dy = d(yzdx) \ .$$

Thus, the generator for

$$\pi_2^* \alpha - \pi_1^* p_{(231)}^* \alpha = ZdX \wedge dY - xdy \wedge dz \ ,$$

$\Lambda_{0,0}$	Adx	Bdy
CdX	$A(x,y,X)\ ,\quad C(x,X,Y)$	$B(x,y,X)\ ,\quad C(y,X,Y)$
	$z = \partial_y A$	$z = -\partial_x B$
	$\partial_X A = \partial_x C$	$\partial_X B = \partial_y C$
	$Z = -\partial_Y C$	$Z = -\partial_Y C$
	$\frac{\partial X}{\partial z} \neq 0\ ,\quad \frac{\partial x}{\partial Z} \neq 0$	$\frac{\partial X}{\partial z} \neq 0\ ,\quad \frac{\partial y}{\partial Z} \neq 0$
DdY	$A(x,y,Y)\ ,\quad D(x,X,Y)$	$B(x,y,Y)\ ,\quad D(y,X,Y)$
	$z = \partial_y A$	$z = -\partial_x B$
	$\partial_Y A = \partial_x D$	$\partial_Y B = \partial_y D$
	$Z = \partial_X D$	$Z = \partial_X D$
	$\frac{\partial Y}{\partial z} \neq 0\ ,\quad \frac{\partial x}{\partial Z} \neq 0$	$\frac{\partial Y}{\partial z} \neq 0\ ,\quad \frac{\partial y}{\partial Z} \neq 0$

TABLE 1. Four basic generating forms with respect to $\tilde{\alpha} = \alpha = zdx \wedge dy$. Shown are the independent variables for each function, the three implicit mapping equations, and the two twist conditions.

becomes

$$\Lambda_{(231),0} = \Lambda_{0,0} + xzdy\ .$$

For example, to reproduce the results of §4 we select the dy and dY components for Λ, so we begin with the B-D form for $\Lambda_{0,0}$ to obtain

$$\Lambda_{(231),0} = (B(x,y,Y) + xz)dy + D(y,X,Y)dY = \hat{A}(y,z,Y)dy + D(y,X,Y)dY\ .$$

As indicated, the mapping equation $B_x = -z$, becomes a new consistency condition: $\partial_x \hat{A} = 0$. The consistency condition $\partial_z B = 0$ becomes a new mapping equation

$$\partial_z \hat{A} = \partial_z B + x = x\ .$$

The remaining two equations are unchanged, reproducing the system (15). Alternatively, the permutation can also directly be applied to the labels (x,y,z) in Tbl. 1 to transform the entire table into that for $\Lambda_{(231),0}$, see Tbl. 2. Note that the twist conditions for the generated maps are geometrically distinct.

Similar tables are easily constructed for the remaining permutations to give a total of thirty-six different generating forms.

$\Lambda_{(231),0}$	Ady	Bdz
CdX	$A(y,z,X)\ ,\quad C(y,X,Y)$	$B(y,z,X)\ ,\quad C(z,X,Y)$
	$x = \partial_z A$	$x = -\partial_y B$
	$\partial_X A = \partial_y C$	$\partial_X B = \partial_z C$
	$Z = -\partial_Y C$	$Z = -\partial_Y C$
	$\frac{\partial X}{\partial x} \neq 0\ ,\quad \frac{\partial y}{\partial Z} \neq 0$	$\frac{\partial X}{\partial x} \neq 0\ ,\quad \frac{\partial z}{\partial Z} \neq 0$
DdY	$A(y,z,Y)\ ,\quad D(y,X,Y)$	$B(y,z,Y)\ ,\quad D(z,X,Y)$
	$x = \partial_z A$	$x = -\partial_y B$
	$\partial_Y A = \partial_y D$	$\partial_Y B = \partial_z D$
	$Z = \partial_X D$	$Z = \partial_X D$
	$\frac{\partial Y}{\partial x} \neq 0\ ,\quad \frac{\partial y}{\partial Z} \neq 0$	$\frac{\partial Y}{\partial x} \neq 0\ ,\quad \frac{\partial z}{\partial Z} \neq 0$

TABLE 2. Four basic generating forms with respect to $\tilde{\alpha} = zdx \wedge dy$ and $\alpha = xdy \wedge dz$.

As an example of the forms shown in Tbl. 2, consider the B-C type generating form

$$\Lambda = (-yX + g(y,z) - h(X,z))dz + (-zY - k(X,Y))dX \ .$$

The generated map is

$$\begin{aligned} X &= x + g_y(y,z) \ , \\ Y &= y + h_X(X,z) \ , \\ Z &= z + k_Y(X,Y) \ . \end{aligned} \tag{20}$$

The twist conditions are trivially satisfied: $\partial_x X(x,y,z) = 1$ and $\partial_Z z(X,Y,Z) = 1$. The much-studied ABC-map has this form [10, 27]. The map (20) is the composition of three, exact volume-preserving shears, e.g., maps of the form $(X,Y,Z) = (x + F(y,z), y, z)$. It is also a first-order volume-preserving integrator of the incompressible flow with vector field $(g_y(y,z), h_x(x,z), k_y(x,y))$ [26].

7. **Some generating forms on $\mathbb{R}^n$.** In this section we construct a generating form for $M = \mathbb{R}^n$, choosing—for simplicity,

$$\begin{aligned} \pi_1^*\alpha &= (-1)^{n-1} x_n dx_1 \wedge \cdots \wedge dx_{n-1}, \\ \pi_2^*\tilde{\alpha} &= X_1 dX_2 \wedge \cdots \wedge dX_n \ . \end{aligned}$$

Here we use the coordinates $(x_1, \ldots, x_n, X_1, \ldots, X_n) \in N = M \times M$. This choice will reproduce formulas that, as far as we know, first appeared in [6].

The form Λ will depend upon the $n-2$ variables $(x_1, x_2, \ldots x_{n-1}, X_2, X_3, \ldots, X_n)$. To develop the notation for this form, define the projections $h_k : N \to M$ by

$$h_k(x_1, \ldots, x_n, X_1, \ldots, X_n) = (x_1, \ldots, x_k, X_{k+1}, \ldots, X_n),$$

for each $k = 1, \ldots, n-1$. Similarly for each k, define the $(n-2)$-form on M

$$\rho_k \equiv \Phi^k \, dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+2} \wedge \cdots \wedge dx_n$$

where $\Phi^k \in C^2(M, \mathbb{R})$. Notice that $h_k^*\rho_k$ is an $(n-2)$-form defined on N.

Theorem 7.1. *Let $\Phi^1, \ldots, \Phi^n$ be C^2 functions on M. Assume that there exist two constants $\ell_1, \ell_2 > 0$ such that, for all $k = 1, \ldots, n-1$ and all $m \in M$, one has*

$$0 < \ell_1 \le \left| \partial_{k,k+1} \Phi^k(m) \right| \le \ell_2 \ .$$

Then, the $(n-2)$-form $\Lambda = \sum_{k=1}^{n-1} h_k^*\rho_k$ is a generating form and the generated map $(X_1, \ldots, X_n) = f(x_1, \ldots, x_n)$ is implicitly given by the n equations

$$\begin{aligned} X_1 &= \partial_2 \Phi^1(x_1, X_2, \ldots, X_n) \ , \\ \partial_k \Phi^k(x_1, \ldots, x_k, X_{k+1}, \ldots, X_n) &= \partial_{k+2} \Phi^{k+1}(x_1, \ldots, x_{k+1}, X_{k+2}, \ldots, X_n) \ , \\ \partial_{n-1} \Phi^{n-1}(x_1, \ldots, x_{n-1}, X_n) &= x_n \ , \end{aligned} \tag{21}$$

for $k = 1, \ldots, n-2$.

Proof. This is a straightforward computation. The differentials of the basic forms are

$$\begin{aligned} d\rho_k =& (-1)^{k-1} \left(\partial_k \Phi^k \right) dx_1 \wedge \cdots \wedge dx_k \wedge dx_{k+2} \wedge \cdots \wedge dx_n \\ &+ (-1)^{k-1} \left(\partial_{k+1} \Phi^k \right) dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n \ . \end{aligned}$$

This implies that, as a form on $N = M \times M$, Λ satisfies:

$$d\Lambda = \sum_{k=1}^{n-1} (-1)^{k-1} \left[\left(\partial_k \Phi^k \circ h_k\right) dx_1 \wedge \cdots \wedge dx_k \wedge dX_{k+2} \wedge \cdots \wedge dX_n \right.$$
$$\left. + \left(\partial_{k+1} \Phi^k \circ h_k\right) dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dX_{k+1} \wedge \cdots \wedge dX_n \right] .$$

Rearranging the terms in the sum we find that

$$\begin{aligned} d\Lambda &= \left(\partial_2 \Phi^1 \circ h_1\right) dX_2 \wedge \cdots \wedge dX_n + (-1)^n \left(\partial_{n-1} \Phi^{n-1} \circ h_{n-1}\right) dx_1 \wedge \cdots \wedge dx_{n-1} \\ &+ \sum_{k=1}^{n-2} (-1)^k \left(\partial_{k+2} \Phi^{k+1} \circ h_{k+1} - \partial_k \Phi^k \circ h_k\right) dx_1 \wedge \cdots \wedge dx_k \wedge dX_{k+2} \wedge \cdots \wedge dX_n .\end{aligned}$$

Therefore, in order to satisfy

$$\left(\pi_2^* \tilde{\alpha} - \pi_1^* \alpha - d\Lambda\right)(x_1, \ldots, x_n, X_1, \ldots, X_n) = 0 ,$$

one needs to have, for $k = 1, \ldots, n-2$,

$$\begin{aligned} X_1 - \partial_2 \Phi^1 \circ h_1 &= 0 , \\ \partial_{k+2} \Phi^{k+1} \circ h_{k+1} - \partial_k \Phi^k \circ h_k &= 0 , \\ x_n - \partial_{n-1} \Phi^{n-1} \circ h_{n-1} &= 0 . \end{aligned} \tag{22}$$

Equations (21) and (22) are the same. The conditions on the functions Φ^k imply that we can solve for $(X_1, \ldots, X_n)$ in terms of $(x_1, \ldots, x_n)$ and vice versa. □

For example, on $\mathbb{R}^3$ with coordinates (x, y, z), this reduces to the generating form

$$\Lambda_{0,(231)} = \Phi^1(x, Y, Z) dZ + \Phi^2(x, y, Z) dx ,$$

which is a permuted version of the A-D form in Tbl. 1.

8. **Generators on other manifolds.** Though local generating forms can be defined for any manifold, when M has nontrivial cohomology the volume form Ω may not be exact and an $(n-1)$-form α may not exist globally. Even if α can be defined on M, there might not be sufficient freedom to obtain a well-defined global generator Λ on $M \times M$. However in this case, it may be possible to define α on a cover of M; e.g., if the local diffeomorphism $p : C \to M$ is a cover, then the volume form $p^*\Omega$ on C may be exact. The point is to choose the cover so that it has trivial n and $n-1$ dimensional cohomology groups.

We would like to remark that our generating forms, by assumption, are global in nature. It would be interesting to find some local conditions that would imply the existence of global generating forms. In the symplectic case, some results in this direction can be found in [14, p. 95]. We leave this issue for future work.

We will suppose that p corresponds to the universal cover of M, and will construct Λ on the product space for the universal cover $N = C \times C$. If Λ is a generating form on N, then it generates a map $g : C \to C$. This map will be the lift of a map $f : M \to M$ if

$$p \circ g = f \circ p .$$

We will show that for this to occur it is sufficient that Γ be invariant under an extension of the group of deck transformations of p to the space $C \times C$.

Recall that the group of deck transformations of a cover p is $T = \{t : C \to C \mid p \circ t = p\}$. Obviously $p \times p : C \times C \to M \times M$ is a cover of $M \times M$ and the group of deck transformations on $C \times C$ is

$$U = \{(t_1, t_2) : C \times C \to C \times C \mid t_1, t_2 \in T\} ;$$

indeed, if $(t_1, t_2) \in U$ then $(p, p) \circ (t_1, t_2) = (p, p)$. We will say that a subgroup Δ of U is of *diagonal type* if it is of the form

$$\Delta = \{(t, \psi(t)) \in U \mid t \in T\}$$

for some fixed group automorphism $\psi : T \to T$. For instance, if we take $\psi = id_T$, then Δ is the diagonal. In general, Δ is isomorphic to the original group T. Invariance of Λ under Δ implies that its generated map is a lift.

Lemma 8.1. *Suppose that C is a cover of M, α and $\tilde{\alpha}$ are $n-1$ forms in C such that $p^*\Omega = d\alpha = d\tilde{\alpha}$, and Λ is a generator with respect to $(\alpha, \tilde{\alpha})$ of an exact volume-preserving map g on C. Then, if $\Gamma = \pi_2^*\tilde{\alpha} - \pi_1^*\alpha - d\Lambda$ is invariant under a subgroup of the deck transformations of $C \times C$ of diagonal type, g is the lift of a volume-preserving map f on M. In addition, if Ω is exact, then f is exact volume-preserving.*

Proof. By assumption Γ vanishes at $(c, g(c)) \in N$ for each $c \in C$. Since Γ is invariant under Δ, $(t, \psi(t))^*\Gamma = \Gamma$. Therefore Γ also vanishes at $(t(c), \psi(t)(g(c)))$. Equivalently,

$$g \circ t = \psi(t) \circ g .$$

Consequently, the two points $t(c)$ and c, which project to the same point $m = p(c)$, have the same projected image $p(g(t(c)) = p(g(c))$. But this implies that $f(m)$ is uniquely defined and satisfies $p \circ g = f \circ p$.

Now $p^*\Omega$ is a volume form on C and g preserves $p^*\Omega$. Hence

$$p^* f^*\Omega = (f \circ p)^*\Omega = (p \circ g)^*\Omega = p^*\Omega.$$

Since p is a local diffeomorphism, we conclude that $f^*\Omega = \Omega$. When the original volume form is exact, the same argument can be used to show that f is also exact. ☐

Note that even when Γ is invariant under Δ, the form Λ need be, and thus may not have a well-defined projection on $M \times M$. Nevertheless, the projected map is well-defined whenever Γ is invariant under a suitable subgroup Δ.

Example. A simple example corresponds to the two-dimensional, generalized standard map

$$f(x, y) = (x + y - V'(x), y - V'(x)) . \tag{23}$$

where $V(x+1) = V(x)$ is the potential. We can think of this map as being defined on the cylinder $M = \mathbb{T} \times \mathbb{R}$. In this case, the universal cover is $C = \mathbb{R}^2$. The group T of deck transformations is generated by a single transformation: $T = \langle\phi_1\rangle$, where $\phi_1(x, y) = (x + 1, y)$. Using the volume form $\Omega = dy \wedge dx$, we may select $\alpha = \tilde{\alpha} = ydx$, and obtain the generating form

$$\Lambda = \frac{1}{2}(X - x)^2 - V(x) , \tag{24}$$

which is a zero-form on $C \times C$, but not on $M \times M$. On the manifold $C \times C$, we will use the diagonal extension Δ of T that consists of the transformations $u(x, y, X, Y) = (x + k, y, X + k, Y)$ for integer k. In other words, we simply use $\psi = id_T$ in the argument above. It is enough to check that the form $\Gamma = YdX - ydx - d\Lambda$ is invariant under (ϕ_1, ϕ_1). Thus the generated map projects to an exact volume-preserving map on $\mathbb{T} \times \mathbb{R}$.

We can also think of (23) as acting on $M = \mathbb{T}^2$. In this case $dy \wedge dx$ is closed, but not exact: $\alpha = ydx$ is not a form on M. The universal cover is still $\mathbb{R}^2$; however, the deck transformation group is now $T = \{t(x, y) = (x + m, y + n) \mid m, n \in \mathbb{Z}\}$ which is generated by two transformations: $T = \langle \phi_1, \phi_2 \rangle$, where

$$\phi_1(x, y) = (x + 1, y) ,$$
$$\phi_2(x, y) = (x, y + 1) .$$

The zero-form (24) is still a generator on the cover; however, the one form

$$\Gamma = Y dX - ydx - d\Lambda = (Y - X + x)dX - (y - X + x + V'(x))dx$$

is not invariant under the trivial diagonal extension of T. Instead, define a different subgroup Δ by choosing the automorphism ψ so that $\psi(\phi_1) = \phi_1$ and $\psi(\phi_2) = \phi_1 \circ \phi_2 = \phi_2 \circ \phi_1 \equiv \phi_3$. Now, the subgroup Δ is generated by the two transformations

$$u_1(x, y, X, Y) = (\phi_1, \phi_1)(x, y, X, Y) = (x + 1, y, X + 1, Y) ,$$
$$u_2(x, y, X, Y) = (\phi_2, \phi_3)(x, y, X, Y) = (x, y + 1, X + 1, Y + 1) .$$

The form Γ is invariant under these deck transformations: $u_1^*\Gamma = \Gamma$ and $u_2^*\Gamma = \Gamma$. Thus every map $g : \mathbb{R}^2 \to \mathbb{R}^2$ generated in this way has the symmetries:

$$g \circ \phi_1 = \phi_1 \circ g ,$$
$$g \circ \phi_2 = \phi_3 \circ g .$$

Therefore g is the lift of a well-defined map of $\mathbb{T}^2$.

9. **One-action maps.** An action-angle map f acts on the manifold $M = \mathbb{T}^d \times \mathbb{R}^k$, having d angle variables, $\theta \in \mathbb{T}^d$, and k action variables, $z \in \mathbb{R}^k$. As an example, consider the generalization of (23)

$$f(\theta, z) = (\theta + \rho(z + F(\theta)), z + F(\theta)) \tag{25}$$

for a "rotation vector" $\rho : \mathbb{R}^k \to \mathbb{T}^d$ and "force" $F : \mathbb{T}^d \to \mathbb{R}^k$. Examples with two angles and one action have been much studied, cf. [10, 7]. Here we will consider this case, setting $d = 2$ and $k = 1$ and

$$\Omega = dz \wedge d\theta_1 \wedge d\theta_2 , \quad \alpha = zd\theta_1 \wedge d\theta_2 . \tag{26}$$

The map (25) is volume preserving. It is easiest to see this by noting that $f = f_1 \circ f_2$ for the volume-preserving shears $f_1(\theta, z) = (\theta + \rho(z), z)$ and $f_2(\theta, z) = (\theta, z + F(\theta))$. Moreover, the map f_1 is always exact, it satisfies (3) with $\lambda_1 = i_W\, d\theta_1 \wedge d\theta_2 = W_1 d\theta_2 - W_2 d\theta_1$ with $W_i = z\rho_i - \int \rho_i dz$. By contrast, f_2 is α-exact only when

$$\int_{\mathbb{T}^2} F(\theta)\, d\theta_1 \wedge d\theta_2 = 0 .$$

This is true if there is a vector field $G : \mathbb{T}^2 \to \mathbb{R}^2$ such that $\nabla \cdot G = F$. In this case, $\lambda_2 = i_G d\theta_1 \wedge d\theta_2$. Finally, the form λ for f is defined using (4).

A *rotational* torus is a two-dimensional torus homotopic to the zero section $\{(0, \theta) \mid \theta \in \mathbb{T}^2\}$. The *net flux* crossing a rotational torus $\mathcal{T}$ is the difference between the volume "below" $f(\mathcal{T})$ and that below $\mathcal{T}$:

$$\mathcal{F}(\mathcal{T}) = \int_{\mathcal{T}} f^*\alpha - \alpha .$$

When f is exact, $\mathcal{F}(\mathcal{T}) = 0$. A consequence is that $f(\mathcal{T}) \cap \mathcal{T} \neq \emptyset$, the so-called intersection property.

The natural integrable case of (25) corresponds to $F = 0$,

$$f(\theta, z) = (\theta + \rho(z), z) \ . \tag{27}$$

For this map, the phase space is foliated by invariant two-tori. Interestingly under some conditions on ρ, a version of KAM-theory can be applied to this system to imply that a Cantor-set of these tori are preserved when f is smoothly perturbed, but remains exact volume-preserving [28, 31].

Indeed, exactness is a necessary requirement for the existence of rotational invariant tori. Suppose that $\mathcal{T}$ and $\hat{\mathcal{T}}$ are rotational tori and f is volume-preserving, then the volume contained between them, $\Delta V = \int_{\hat{\mathcal{T}}} \alpha - \int_{\mathcal{T}} \alpha$, is invariant. This implies that the flux $\mathcal{F}$ is independent of the choice of torus. Consequently if f has an invariant torus then its net flux must vanish, $\mathcal{F} = 0$. Since $\mathcal{F}(\mathcal{T}) = 0$ for any rotational torus, $f^*\alpha - \alpha$ must be exact. Therefore a necessary condition for the existence of rotational invariant tori is that f be exact volume-preserving.

When $M = \mathbb{T}^2 \times \mathbb{R}$, there is only one choice for α that will make the standard volume form exact, (26). Thus, we will consider a generating equation of the form

$$\begin{aligned} \Gamma &= \pi_2^*\alpha - \pi_1^*\alpha - d\Lambda \\ &= Z d\Theta_1 \wedge d\Theta_2 - z d\theta_1 \wedge d\theta_2 - d\Lambda \ . \end{aligned}$$

Taking $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we can define the generating form Λ on the universal cover, $C = \mathbb{R}^3$, of M. Using (θ, z, Θ, Z) as coordinates on $\mathbb{R}^3 \times \mathbb{R}^3$, a suitable generating form is

$$\Lambda = [(\Theta_1 - \theta_1)Z - \Phi(\theta_1, \Theta_2, Z)]\, d\Theta_2 - [(\Theta_2 - \theta_2)z - \Psi(\theta_1, z, \Theta_2)]\, d\theta_1. \tag{28}$$

The appropriate diagonal extension of the group of deck transformations of C consists of the transformations

$$u_k(\theta, z, \Theta, Z) = (\theta + k, z, \Theta + k, Z) \ , \quad k \in \mathbb{Z}^2 \ .$$

In order for Λ to generate the lift of a map on M, it must be invariant: $u_k^*\Lambda = \Lambda$. This requirement is easily seen to be satisfied by (28) when Φ and Ψ are periodic in each of their angular arguments.

The differential of the generating form (28) is

$$\begin{aligned} d\Lambda = &(z - Z - \partial_{\theta_1}\Phi - \partial_{\Theta_2}\Psi)\, d\theta_1 \wedge d\Theta_2 \\ &+ (\Theta_1 - \theta_1 - \partial_Z\Phi)\, dZ \wedge d\Theta_2 + (\theta_2 - \Theta_2 + \partial_z\Psi)\, dz \wedge d\theta_1 \\ &+ Z d\Theta_1 \wedge d\Theta_2 - z d\theta_1 \wedge d\theta_2 \ . \end{aligned}$$

Note that in this case $d\Lambda$ includes the terms in $\pi_2^*\alpha - \pi_1^*\alpha$. The remaining three terms in $d\Lambda$ must vanish, and this determines the generated map

$$\begin{aligned} \Theta_1 &= \theta_1 + \partial_Z\Phi(\theta_1, \Theta_2, Z) \ , \\ \Theta_2 &= \theta_2 + \partial_z\Psi(\theta_1, z, \Theta_2) \ , \\ Z &= z - \partial_{\theta_1}\Phi(\theta_1, \Theta_2, Z) - \partial_{\Theta_2}\Psi(\theta_1, z, \Theta_2) \ . \end{aligned}$$

This map is one-to-one when $\partial_{z\Theta_2}\Psi \neq 1$ and $\partial_{\theta_1 Z}\Phi \neq -1$. This map will have the form of a perturbation of the integrable map (27) if we set

$$\begin{aligned} \Phi &= H_1(Z) + \epsilon F(\theta_1, \Theta_2) \ , \\ \Psi &= H_2(z) + \epsilon G(\theta_1, \Theta_2) \ , \end{aligned}$$

which gives the semi-explicit map

$$\begin{aligned}\Theta_1 &= \theta_1 + \partial_Z H_1(Z)\ , \\ \Theta_2 &= \theta_2 + \partial_z H_2(z)\ , \\ Z &= z - \epsilon[\partial_{\theta_1} F(\theta_1, \Theta_2) + \partial_{\Theta_2} G(\theta_1, \Theta_2)]\ .\end{aligned}$$

10. **Conclusions.** We have shown that exact volume-preserving maps on an n-dimensional manifold can have implicit generating $(n-2)$-forms in the same way that symplectic maps can have implicit generating functions (zero forms). In both cases, the generated maps must satisfy certain necessary geometrical conditions that we have called *twist conditions*. For the n-dimensional case, there are $n-1$ such conditions. It would be interesting to characterize the exact volume-preserving diffeomorphisms that can be generated in this way in terms of a suitable set of twist conditions.

One of the reasons for defining generating functions is to obtain variational principles. These are used to great effect, for example, in Aubry-Mather theory for area-preserving twist maps [24]. Variational principles for exact incompressible vector fields have been studied by Gaeta and Morano [12] (where they were called "globally Liouville vector fields"). It would be interesting to extend their analysis to the map case.

Another possible use for generating forms is as integration algorithms for incompressible flows. Implicit generators are commonly used in symplectic integration algorithms [26].

Generating functions for symplectic maps are also used to compute the symplectic area of lobes in the theory of transport. We will generalized this result to the exact volume-preserving case in a forthcoming paper [21].

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E-mail address: `lomeli@itam.mx`
E-mail address: `James.Meiss@colorado.edu`