CLASSIFICATION OF THE UMBILIC POINT IN IMMISCIBLE THREE-PHASE FLOW IN POROUS MEDIA

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Abstract. We consider the flow in a porous medium of three fluids that do not mix nor interchange mass. Under simplifying assumptions this is the case for oil, water and gas in a petroleum reservoir. For a simple geometry, the horizontal displacement of a pre-existent uniform mixture by another injected mixture gives rise to a Riemann problem for a system of two conservation laws. Such a system depends on laboratory-measured relative permeability functions for each of the three fluids. For Corey models each permeability depends solely on the saturation of the respective fluid, giving rise to systems containing an umbilic point in the interior of the saturation triangle. It has been conjectured that the structure of the Riemann solution in the saturation triangle is strongly influenced by the nature of the umbilic point, which is determined by the quadratic expansion of the flux function nearby. In 1987 it was proved that, for very general Corey permeabilities, umbilic points have types I or II in Schaeffer&Shearer's classification.

In the current work we find precisely the boundaries where the transition occurs in the saturation triangle, which was not done in 1987. The novel tool is a constructive method for determining type I.

1. **Introduction.** In this work, we classify a special hyperbolic singularity, called *umbilic point*, which arises in state space for a class of conservation laws that describe oil recovery; the class is derived in Section 4. The umbilic point is classified according to Schaeffer and Shearer scheme, [8], which applies to systems of two conservation laws with fluxes that are well approximated, in some sense, by quadratic fluxes. In Appendix of [8] Schaeffer, Shearer, Marchesin and Paes-Leme, showed that the umbilic point of strongly convex permeability models has type I or type II. However, they did not determine parameters where the change from type I to II occurs. In Section 3, we detect the transition by means of a new simple way to identify type I singularities based on the quadratic expansion of the flux functions in

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the conservation laws about the umbilic point. The classification is given in Section 5 and depends on the sign of a quartic function of two variables. Surprisingly, the quartic function turns out to be a product of four affine functions of two variables. Therefore, the change of sign occurs on straight lines and the fluxes are type I inside a small triangle in state space, and type II outside.

2. Background.

Consider a 2×2 system of conservation laws in one space variable

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial x}F(U) = 0, \tag{1}$$

where $-\infty < x < \infty$, $t \geq 0$, $U = U(x,t) \in \mathcal{D}$, $F : \mathcal{D} \to \mathbb{R}^2$ is a C^2 function and $\mathcal{D}$ is an open set of $\mathbb{R}^2$. Equation (1) together with the initial conditions

$$U(x,0) = \begin{cases} U_L \text{ if } x < 0 \\ U_R \text{ if } 0 < x \end{cases} \tag{2}$$

are the so called Riemann problem.

Both (1) and (2) may be rescaled by $(x,t) \to (cx, ct)$ for $c > 0$. Therefore, the solutions of (1) and (2) should depend only on $\xi = x/t$, thus, they are sequences of constant states and two kinds of solutions: rarefactions and shocks.

A rarefaction, also called rarefaction fan, is a solution of

$$(DF(U) - \xi I)\, U_\xi = 0,$$

for ξ in an interval. If (1) is strictly hyperbolic the eigenvalues of $DF(U)$ are ordered, $\lambda_1(U) < \lambda_2(U)$. Hence, assuming strictly hyperbolicity, there are two kinds of rarefactions, each one associated to an eigenpair.

A shock is a discontinuous solution that moves with speed s along which U jumps form a left state U_- to a right state U_+,

$$\lim_{x\to st^-} U(x,t) = U_-\ , \ \lim_{x\to st^+} U(x,t) = U_+,$$

and the well known Rankine-Hugoniot condition holds:

$$F(U_+) - F(U_-) - s\,(U_+ - U_-) = 0. \tag{3}$$

There are only three kinds of relevant shocks in [8], nonetheless, we define four kinds following [6, 7], which extended some results of [8].

Definition 2.1. A solution of (3) is a:
(i) *slow shock* iff $\lambda_1(U_+) < s < \lambda_1(U_-)$ and $s < \lambda_2(U_+)$;
(ii) *fast shock* iff $\lambda_2(U_+) < s < \lambda_2(U_-)$ and $\lambda_1(U_-) < s$;
(iii) *overcompressive shock* iff $\lambda_2(U_+) < s < \lambda_1(U_-)$.
(iv) *crossing discontinuity* iff $\lambda_1(U_-) < s < \lambda_2(U_-)$ and $\lambda_1(U_+) < s < \lambda_2(U_+)$.

The oriented integral curves of each eigenvector field form *rarefaction curves* in state space. The rarefaction curves have the orientation of increasing eigenvalue. The solutions of (3) for a fixed U_- that satisfy any of the condition of Definition 2.1 form *shock curves* in state space.

We give a definition of umbilic point slightly different from the one given in [8] by Schaeffer and Shearer.

Definition 2.2. Let F be a C^2 function on an open set $\mathcal{D} \subseteq \mathbb{R}^2$, $F : \mathcal{D} \mapsto \mathbb{R}^2$, $U \mapsto F(U)$ and DF its derivative. Assume that DF has two equal eigenvalues and is diagonalizable at U^*. If there exists a neighborhood $\mathcal{N}$ of U^* such that DF has distinct real eigenvalues for all $U \in \mathcal{N} \setminus U^*$ then U^* is an *umbilic point* of F.

Instead of saying that U^* is an umbilic point of Eq. (1) with flux F, we will say just that U^* is an *umbilic point of* F. (For a singularity where $\mathcal{N}$ intersects the hyperbolic and elliptic regions see [2].) The following simplifications are assumed.

Simplification 2.3. Let F and U^* be as in Definition 2.2, then:
(i) The flux F has no constant terms;
(ii) The umbilic point is the origin, $U^* = (0, 0)$;
(iii) The derivative $DF(U^*)$ is the null matrix.

The Simplification 2.3 holds without loss of generality since: (i) constant terms do not change the solution of conservation laws, thus they may be neglected; (ii) performing the translation $U \to U - U^*$ the umbilic point lies at the origin; (iii) changing the inertial frame, $(x, t) \to (x - \xi^* t, t)$, where ξ^* is the eigenvalue of $DF(U^*)$, the derivative becomes $DF(U^*) - \xi^* I = 0$.

The hypothesis $\mathcal{H}$ of [8] motivates the following definition.

Definition 2.4. Let F and U^* be as in Definition 2.2 and G be the second order Taylor expansion of F around U^*. If U^* is an umbilic point of G then F is an *$\mathcal{H}$-flux*.

We remark that if U^* is an umbilic point of the quadratic function G then it is also an umbilic point of F. However, the inverse implication is false.

Henceforth, F will be an $\mathcal{H}$-flux, G will be the second order Taylor expansion of F (thus G is an $\mathcal{H}$-flux too) and U^* will be the umbilic point of F and G.

Remark 1. Since we assume the Simplification 2.3 has been made on F, G is an homogeneous quadratic $\mathcal{H}$-flux.

Schaeffer and Shearer in [8] classified generic $\mathcal{H}$-fluxes depending on the number and type of shock and rarefaction curves through the umbilic point. They did so in two steps. First, they established four robust topological configurations for homogeneous quadratic $\mathcal{H}$-fluxes (see Theorem 2.7 below). Then, for non degenerate homogeneous quadratic $\mathcal{H}$-flux, they proved the higher order terms do not affect the classification (see Theorem 2.10 below).

Palmeira and Marchesin, [6, 7], proved that higher order terms also do not affect the topological behavior of the shock and rarefaction curves in a neighborhood of U^* (not only through U^*) – see Theorem 2.11 below. This result arises in the context of the *Wave Manifold*, see [4].

Definition 2.5. The fluxes G_1 and G_2 are equivalent if and only if there is a constant invertible 2×2 matrix M such that

$$G_1(U) = M^{-1} G_2(MU).$$

Lemma 2.6. *Equivalence preserves the structure of the shock and rarefaction curves. That is, the mapping $U \mapsto M^{-1}U$ maps shock and rarefaction curves to shock and rarefaction curves, respectively.*

A version of the main result of Schaeffer and Shearer is presented in the following Theorems 2.7 and 2.10.

Theorem 2.7. *Every G is equivalent to the following normal flux*

$$Q(u, v)^T = \begin{bmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{bmatrix} \tag{4}$$

with $a \neq b^2 + 1$. There are four different robust configurations for the shock and rarefaction curves of G through U^*, labeled from I to IV. The flux G has:
(i) type I if and only if $a < \frac{3}{4}b^2$;
(ii) type II if and only if $\frac{3}{4}b^2 < a < b^2 + 1$;
(iii) type III if and only if $b^2 + 1 < a < \Phi(b)$; [1]
(iv) type IV if and only if $\Phi(b) < a$.

Definition 2.8. The quadratic flux function G is degenerate if it is equivalent to Q with $a = \frac{3}{4}b^2$ or $a = \Phi(b)$.

We remark that if $a = b^2 + 1$ then the Q flux has no umbilic point, rather it has a straight line of points with just one real eigenvalue. Thus, if $a = b^2 + 1$, Q is not a $\mathcal{H}$-flux and no G may be equivalent to Q. (Schaeffer and Shearer define degenerate Q instead of degenerate G; Q is degenerate also if $a = b^2 + 1$.)

The point (i) of the following Definition is natural, the point (ii) will be useful:

Definition 2.9. A given F is classified depending on its corresponding G:
(i) if G has type I to IV then F has type I to IV, respectively;
(ii) if G is a border case between type I and type II (*i.e.*, if G is equivalent to Q with $a = \frac{3}{4}b^2$) then F is of border-type I/II;

The classification of Schaeffer and Shearer depends on the shock and rarefaction curves through U^*.

Theorem 2.10. *If G is non degenerate, the shock and rarefaction curves of G through U^* are in one-to-one correspondence with shock and rarefaction curves of F through U^*. Indeed every shock and rarefaction curve of F is tangent at U^* to a shock and rarefaction curve of G.*

Despite the fact that the classification depends only on the shock and rarefaction curves through U^*, there is a one-to-one correspondence with shock and rarefaction curves of F around U^*. The following Theorem summarizes the stability results from [6, 7].

Theorem 2.11. *Shock and rarefaction curves near the umbilic point are structurally stable under C^3 perturbations of fluxes in the Whitney topology.*

We remark that rarefactions curves are also structurally stable under C^2 perturbations of fluxes.

3. A constructive classification of type I fluxes.

The work [8] provides the construction for a normal flux equivalent to a given $\mathcal{H}$-flux. Nevertheless, it is not always practical to use directly this construction in order to obtain the normal flux (4). The new Theorem 3.2 allows us to determine if a flux has type I.

For any G we have (recall Remark 1): (i) $DG(U)$ is linear in u and v; (ii) $\det(DG(U))$ is a quadratic form in u and v. These facts motivate the following definition.

Definition 3.1. For a given G as in Definition 2.4 we associate the constant symmetric matrix N_G such that $\det(DG(U)) = U^T N_G U$.

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[1] $\Phi(b)$ is given implicitly by $-32\, b^4 + b^2 \left(27 + 36\,(a-2) - 4\,(a-2)^2\right) + 4\,(a-2)^3 = 0$.

Theorem 3.2. *Let F, G and N_G be as in Definitions 2.2, 2.4 and 3.1, then:*
(i) F has type I if and only if $\det(N_G) > 0$*;*
(ii) F has border-type I/II if and only if $\det(N_G) = 0$*;*
(iii) F has neither type I nor border-type I/II if and only if $\det(N_G) < 0$.

Proof. The derivative of the normal flux Q from Equation (4) is

$$DQ(U) = \begin{bmatrix} 2au + 2bv & 2bu + 2v \\ 2bu + 2v & 2u \end{bmatrix},$$

thus the matrix N_Q associated to DQ is:

$$N_Q = \begin{bmatrix} 4a - 4b^2 & -2b \\ -2b & -4 \end{bmatrix}.$$

Then $det\,(N_Q) = -16a + 12b^2$, thus:
(i) Q has type I if and only if $det\,(N_Q) > 0$;
(ii) Q has border-type I/II if and only if $det\,(N_Q) = 0$;
(iii) Q has neither type I nor border-type I/II if and only if $det\,(N_Q) < 0$.
If Q and G are equivalent then there exists an invertible constant matrix M such that $Q(U) = M^{-1}G(MU)$ for all $U \in \mathbb{R}^2$, thus

$$\begin{aligned} DQ(U) = M^{-1}DG(MU)M &\Rightarrow det(DQ(U)) = det(DG(MU)) \Leftrightarrow \\ \Leftrightarrow U^T N_Q U = U^T M^T N_G M U &\Leftrightarrow N_Q = M^T N_G M \Rightarrow \\ \Rightarrow det(N_Q) = det(M^T N_G M) &\Leftrightarrow det(N_Q) = det^2(M) det(N_G). \end{aligned}$$

Therefore, if G and Q are equivalent $det(N_Q)$ and $det(N_G)$ have the same sign. Since F is classified depending on G, as in Definition 2.9, the proof is complete. □

Remark 2. If G is non degenerate, Theorems 2.10 and 2.11 ensure that shock and rarefaction curves around U^* for F and G are topologically equivalent. If G is degenerate, for instance, F is border-type I/II, then nothing is known about the relation between these shock and rarefaction curves of F and G.

4. **The general immiscible three-phase flow in porous media.** In this section we give a brief derivation of the model we studied, sometimes named Corey with general power permeabilities, which arises in Petroleum Engineering. We consider a one-dimensional, horizontal (*i.e.*, with negligible gravitational effects), incompressible flow in a homogeneous porous media filled with three immiscible phases (*e.g.*, water, oil and gas) with no mass exchange between phases. The fraction of the porous volume occupied by each phase is called saturation and is denoted by s_i, for $i \in \{w, o, g\}$. (The subscripts w, o and g stand for water, oil and gas, respectively.) Admitting no unfilled pore space and no other phase we have $\sum_j s_j = 1$.

Definition 4.1. The domain of (s_w, s_o) such that $s_w + s_o < 1$, $0 < s_w$ and $0 < s_o$ is called the saturation triangle Δ.

We also define: ϕ, the porosity of the medium; ρ_i, the constant density of phase i; v, total seepage velocity of all fluids; f_i, fractional flow of phase i, so $\sum_j f_j = 1$. The mass conservation of each of the three phases is given by:

$$\frac{\partial}{\partial t}(\phi \rho_i s_i) + \frac{\partial}{\partial x}(\rho_i v f_i) = 0, \quad \text{for} \quad i \in \{w, o, g\}. \tag{5}$$

Assuming that v is a non zero constant, determined by boundary conditions, it is possible to eliminate one equation and to nondimensionalize Eq. (5) to obtain

$$\begin{gathered}\frac{\partial}{\partial t}s_i + \frac{\partial}{\partial x}f_i = 0, \quad \text{for} \quad i \in \{w, o\}, \\ s_g = 1 - s_w - s_o \quad \text{and} \quad f_g = 1 - f_w - f_o.\end{gathered} \tag{6}$$

The fractional flow is given by $f_i = \frac{m_i}{\sum m_i}$ where $m_i = b_i s_i^{a_i}$, $a_i > 1$ and $b_i > 0$, is the mobility of each phase. Therefore, we have

$$\begin{aligned} f_w(s_w, s_o) &= \frac{b_w s_w^{a_w}}{b_w s_w^{a_w} + b_o s_o^{a_o} + b_g(1 - s_w - s_o)^{a_g}}, \\ f_o(s_w, s_o) &= \frac{b_o s_o^{a_o}}{b_w s_w^{a_w} + b_o s_o^{a_o} + b_g(1 - s_w - s_o)^{a_g}}. \end{aligned} \tag{7}$$

We will study (1) with the flux function $F : \Delta \to \mathbb{R}^2$, $(s_w, s_o) \mapsto (f_w, f_o)^T$ with f_w and f_o as in Eq. (7) – see Definition 4.1.

5. Classification of the umbilic point for general immiscible three-phase flow in porous media.

In [3] and [8] it was proved that the system (6) with flux (7) has an umbilic point at (s_w, s_o) satisfying

$$\frac{\mathrm{d}m_w(s_w)}{\mathrm{d}s_w} = \frac{\mathrm{d}m_o(s_o)}{\mathrm{d}s_o} = \frac{\mathrm{d}m_g(s_g)}{\mathrm{d}s_g}.$$

We call α, β the coordinates of the umbilic point and $\gamma = 1 - \alpha - \beta$.

In Appendix of [8] Schaeffer, Shearer, Marchesin and Paes-Leme showed, in a more general case, that the flux F of Eq. (7) is type I or II but they did not determine the location of the border between I and II. The main result of our paper (Theorem 5.2) is the classification of F depending on the position of the umbilic point (α, β) on the saturation triangle. To do that we will use Theorem 3.2.

Expanding f_w and f_o of Eq. (7) around (α, β) up to second order, $s_w = u + \alpha$, $s_o = v + \beta$, $(s_g = w + \gamma)$, and performing Simplification 2.3 we obtain a homogeneous quadratic $\mathcal{H}$-flux that allows us to classify the flux F:

$$\begin{aligned} g_w(u, v) &= \frac{q_w(\bar{m}_o + \bar{m}_g) - q_g \bar{m}_w}{\bar{m}^2}\frac{u^2}{2} - \frac{q_g \bar{m}_w}{\bar{m}^2}vu - \frac{(q_o + q_g)\bar{m}_w}{\bar{m}^2}\frac{v^2}{2}, \\ g_o(u, v) &= -\frac{(q_w + q_g)\bar{m}_o}{\bar{m}^2}\frac{u^2}{2} - \frac{q_g \bar{m}_o}{\bar{m}^2}vu + \frac{q_o(\bar{m}_w + \bar{m}_g) - q_g \bar{m}_o}{\bar{m}^2}\frac{v^2}{2}, \end{aligned} \tag{8}$$

where:

$$\begin{array}{lll} \bar{m}_w = b_w \alpha^{a_w}, & \bar{m}_o = b_o \beta^{a_o}, & \bar{m}_g = b_g \gamma^{a_g}, \\ q_w = A_w^2 b_w \alpha^{a_w - 2}, & q_o = A_o^2 b_o \beta^{a_o - 2}, & q_g = A_g^2 b_g \gamma^{a_g - 2}, \\ \bar{m} = \bar{m}_w + \bar{m}_o + \bar{m}_g, & A_i = \sqrt{a_i(a_i - 1)} & \text{for } i \in \{w, o, g\}. \end{array}$$

For simplicity we write $A_w = A$, $A_o = B$ and $A_g = C$. We define $G(u, v) = (g_w(u, v), g_o(u, v))^T$ where $g_w(u, v)$ and $g_o(u, v)$ are given by Eq. (8). The following triangle is used in the classification of F.

Definition 5.1. Let us define the triangle $\mathcal{T}$ in the plane (α, β) with vertices:

$$\left(\tfrac{AB}{AB+BC}, 0\right) \ , \ \left(0, \tfrac{AB}{CA+AB}\right) \ \text{and} \ \left(\tfrac{CA}{BC+CA}, \tfrac{BC}{BC+CA}\right);$$

along the sides of the triangle Δ in Fig. 1. The corresponding γ coordinates are $\frac{BC}{AB+BC}$, $\frac{CA}{CA+AB}$ and 0.

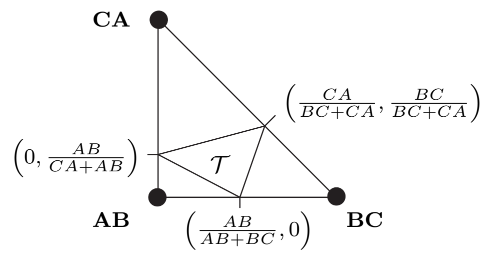

FIGURE 1. The triangle $\mathcal{T}$ inside of Δ and the vertices of $\mathcal{T}$ as centers of mass.

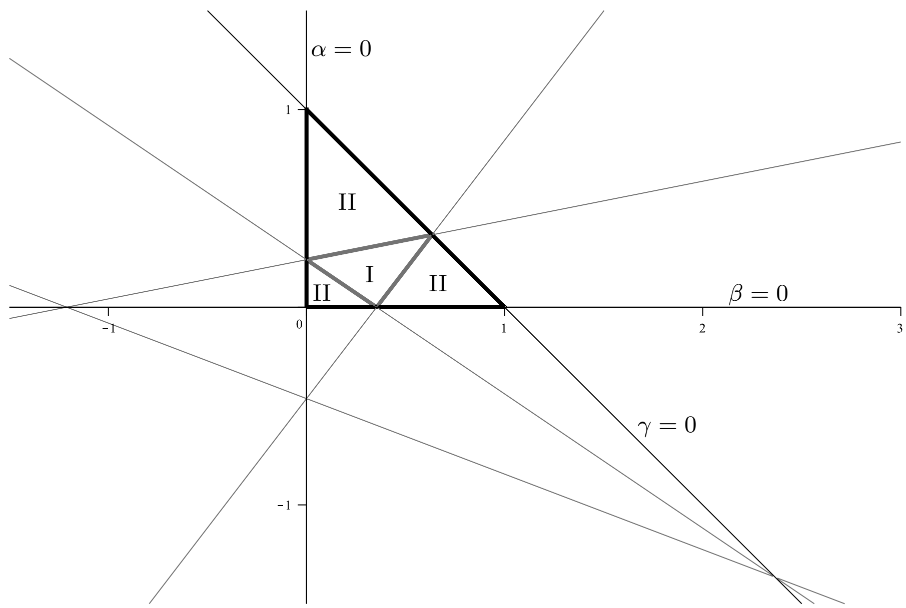

FIGURE 2. The plane (α, β) and the classification on the triangle Δ in type I and II. (Example for $A^2 = 6$, $B^2 = 2$ and $C^2 = 20$.)

Assuming that A, B, C are all different, each straight line crosses each of the three axes $\alpha = 0$, $\beta = 0$ and $\gamma = 0$ once. The intersection points with the axes are the same for each pair of straight lines (see Fig. 2). There exists a curious way to compute $\mathcal{T}$. Let us put masses on the vertices of Δ: (i) a mass BC at $\alpha = 1$; (ii) a mass CA at $\beta = 1$; (iii) a mass of AB at $\gamma = 1$; (see Fig. 1). Then, in order to calculate the vertex of $\mathcal{T}$ that lies at one edge of Δ, we must swap the masses on the vertices of that edge and then calculate the center of mass.

Theorem 5.2. *The flux F of Eq. (7) has:*
(i) type I if and only if (α, β) lies inside $\mathcal{T}$;
(ii) type II if and only if (α, β) lies outside $\mathcal{T}$;
(iii) border-type I/II if and only if (α, β) lies on the edges of $\mathcal{T}$.

Proof. We classify F using Theorem 3.2. The second order expansion of F is G in Eq. (8). Straightforward calculations show the matrix N_G is

$$N_G = \frac{1}{2\bar{m}^3}\begin{bmatrix} -\bar{m}_o q_g q_w & q_w \bar{m}_g q_o - q_g \bar{m}_w q_o - \bar{m}_o q_g q_w \\ q_w \bar{m}_g q_o - q_g \bar{m}_w q_o - \bar{m}_o q_g q_w & -q_g \bar{m}_w q_o \end{bmatrix}.$$

and its determinant $\det(N_G)$ is the product of the negative constant

$$-\frac{4A^2B^2C^2\alpha^2\beta^2\gamma^2\bar{m}^6}{q_w q_o q_g \bar{m}_w \bar{m}_o \bar{m}_g}$$

by the product of four linear functions:

$$\begin{array}{ll} +BC\alpha + AC\beta - AB\gamma; & +BC\alpha - AC\beta + AB\gamma; \\ -BC\alpha + AC\beta + AB\gamma; & +BC\alpha + AC\beta + AB\gamma. \end{array} \tag{9}$$

Therefore, the zero set of $\det(N_G)$ is formed by four straight lines, on which F has border-type I/II. Substituting any of α, β, γ by zero in (9) we find the vertices of $\mathcal{T}$. We may check the sign of $\det(N_G)$ to prove that F has type I only inside of $\mathcal{T}$. Since in [8] it was proved that F has type I or II, it follows that F has type II outside of $\mathcal{T}$. □

In the special case of equal mobilities, $a_w = a_o = a_g$, $A = B = C$, the triangles Δ and $\mathcal{T}$ have parallel edges, the vertices of $\mathcal{T}$ are $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$. The case of expoent 2 is studied by Asakura in [1] and by Isaacson et al. in [5].

6. **Conclusion.** In this work we classify the singularities that arise from a model of Petroleum Engineering. The model depends on parameters, a_i, b_i for $i \in \{w, o, g\}$. The location of the singularity, the umbilic point, depends on the same parameters. However, the subdivision of the saturation triangle in type I or type II umbilic points depends only on the parameters A, B and C; *i.e.*, on the mobility powers a_w, a_o and a_g, in a very geometrical way.

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