ON SINGULAR POINTS FOR CONVEX PERMEABILITY MODELS

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Abstract. We focus on a system of two conservation laws representing a large class of models relevant for petroleum engineering, the domain of which possesses singular points. It has been conjectured that the structure of the Riemann solution in the saturation triangle is strongly influenced by the nature of the umbilic point. In the current work we show that features originally related to umbilic points actually belong to a distinct point, the new Equal-Speed Shocks point.

Even though the location of the umbilic point is known, for the first time, we relate the umbilic point to a physical property, namely, the minimum of the total mobility for any Corey model.

1. **Introduction.** We are interested in the study of injection problems for 2×2 systems of conservation laws; a survey may be found in [2, 4, 7, 12] and references therein. The solution construction for the injection of water and gas is presented in [2] for the case of quadratic Corey models.

We discuss the location of the umbilic point in the interior of the triangle and the new "Equal-Speed Shocks" (ESS) point, which arises in these more general non-symmetric models. Analyses on umbilic points were made in the last few years [8, 9, 14]. The special case of quadratic Corey models is discussed in [1].

We consider models for reservoirs that may contain three fluids, for concreteness, we call them water, gas, and oil; although they could be any three fluids that are immiscible with each other. For simplicity, we assume that the three phases are incompressible, that gravitational segregation and capillary effects are negligible, and that there is no mass transfer among the phases. The flow occurs in one dimension at constant injection rate and fixed proportion of injected fluids. The mobility of each phase is assumed to be a convex function of its own saturation and inversely proportional to the phase viscosity. The mathematical model consists of

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two conservation laws representing Darcy's law combined with mass conservation for two of the phases. The flow problem depends on two viscosity ratios and the precise choice of mobilities. (The overall picture of solutions given in [2] is essentially unchanged in the more general class of models treated in this work, see [5, 6].)

A Corey-type model loses strict hyperbolicity at an umbilic point. Models without umbilic points have been considered for three-phase flow; see [7]. They yield simple solutions for the injection problem. However, they are unrealistic because immiscibility of the three phases seems to be related to loss of strict hyperbolicity [3, 14, 16], *i.e.*, either umbilic points or elliptic regions are present. Models with umbilic points have complicated solutions, but are still well behaved mathematically; see [9, 12] for a review of their properties.

This work is organized as follows. In Sec. 2 the convex permeability models are introduced; in Sec. 2.1.1 we give a brief review of rarefaction fans, shock waves and properties of quadratic Corey models. Section 3 describes certain structures in state space; in Sec. 3.1 we identify features of the umbilic point and in Sec. 3.2 we describe curves with a certain equal shock speed property to the boundaries, the intersection of which is the ESS point. Finally, the conclusions are in Sec. 4.

2. **Mathematical model.**

Consider the flow of a mixture of three fluid phases (which, for concreteness, are called water, gas and oil) in a thin, horizontal cylinder of porous rock. Let $s_{\mathrm{w}}(x,t)$, $s_{\mathrm{g}}(x,t)$ and $s_{\mathrm{o}}(x,t)$ denote the respective saturations at distance x along the cylinder, at time t. Because $s_{\mathrm{w}} + s_{\mathrm{g}} + s_{\mathrm{o}} = 1$ and $0 \leq s_{\mathrm{w}},\, s_{\mathrm{g}},\, s_{\mathrm{o}} \leq 1$, the state space of the fluid mixture is the saturation triangle Δ; see *e.g.* Fig. 1. In our analysis, we choose s_{w} and s_{g} as the two independent variables, thus $S := (s_{\mathrm{w}},\, s_{\mathrm{g}})$; the vertices of Δ are $\mathrm{W} = (1,\, 0)$, $\mathrm{G} = (0,\, 1)$ and $\mathrm{O} = (0,\, 0)$.

2.1. **Conservation laws.**

Three-phase flow in 1D at constant injected rate is governed by the non-dimensionalized system $\partial S/\partial t + \partial F(S)/\partial x = 0$, or

$$\frac{\partial s_{\mathrm{w}}}{\partial t} + \frac{\partial f_{\mathrm{w}}(s_{\mathrm{w}},\, s_{\mathrm{g}})}{\partial x} = 0, \tag{1}$$

$$\frac{\partial s_{\mathrm{g}}}{\partial t} + \frac{\partial f_{\mathrm{g}}(s_{\mathrm{w}},\, s_{\mathrm{g}})}{\partial x} = 0, \tag{2}$$

representing conservation of water and gas. The flow functions $f_{\mathrm{w}}(s_{\mathrm{w}},\, s_{\mathrm{g}})$ and $f_{\mathrm{g}}(s_{\mathrm{w}},\, s_{\mathrm{g}})$ are determined by the relative permeabilities of the three phases.

Although each fluid phase becomes immobile below an residual saturation, for simplicity we assume that the relative permeabilities are strictly positive within the saturation triangle. (In Engineering language, s_{w}, s_{g}, and s_{o} are "reduced saturations".) From Darcy's law the fluxes are

$$f_\alpha(S) = \frac{\mathrm{m}_\alpha(S)}{\mathrm{m}(S)}, \quad \text{for} \quad \alpha = \mathrm{w},\, \mathrm{g},\, \mathrm{o}, \quad \text{where} \quad \mathrm{m} := \mathrm{m}_{\mathrm{w}} + \mathrm{m}_{\mathrm{g}} + \mathrm{m}_{\mathrm{o}} \tag{3}$$

is the total mobility; m_{w}, m_{g}, m_{o} represent the relative mobility of each phase. Each mobility is a ratio between relative permeability and viscosity of the fluid, it is described by the continuous function:

$$\mathrm{m}_\alpha(S) := \frac{k_\alpha(S)}{\mu_\alpha}, \qquad \alpha = \mathrm{w},\, \mathrm{g},\, \mathrm{o}, \tag{4}$$

where μ_α is the given constant viscosity of each phase α.

A Corey type model is defined by a set of mobilities $\mathrm{m}_\alpha(s_\alpha)$ that are nondecreasing continuous functions of their own saturation s_α. In this work we focus on convex Corey models, which obey the following restrictions.

Definition 2.1. A Corey model is said to be **convex** when the mobilities are $\mathcal{C}^1[0,\,1]\cap\mathcal{C}^2(0,\,1)$ functions satisfying:

1. $\mathrm{m}_\alpha(s_\alpha) > 0$ for $s_\alpha \in (0,\,1]$ and $\mathrm{m}_\alpha(0) = 0$,
2. $\mathrm{m}'_\alpha(s_\alpha) > 0$ for $s_\alpha \in (0,\,1]$ and $\mathrm{m}'_\alpha(0) = 0$,
3. $\mathrm{m}''_\alpha(s_\alpha) \geq 0$ for $s_\alpha \in (0,\,1)$,
4. no pair of the quantities $\mathrm{m}''_\mathrm{w}(s_\mathrm{w})$, $\mathrm{m}''_\mathrm{g}(s_\mathrm{g})$, $\mathrm{m}''_\mathrm{o}(s_\mathrm{o})$ vanish simultaneously for any point in the interior of the saturation triangle ($0 < s_\mathrm{w},\, s_\mathrm{g},\, s_\mathrm{o} < 1$).

Remark 1. In the presence of nonzero residual saturations, one can easily formulate an appropriate extension of Definition 2.1.

Remark 2. In this work, for the purpose of illustrating facts with figures, we use the following mobilities:

$$\mathrm{m}_\mathrm{w}(s_\mathrm{w}) = (s_\mathrm{w})^{3.2857}/1, \quad \mathrm{m}_\mathrm{g}(s_\mathrm{g}) = (s_\mathrm{g})^{2.65}/0.5, \quad \mathrm{m}_\mathrm{o}(s_\mathrm{o}) = (s_\mathrm{o})^{5.8357}/2,$$

based on a best fit for homogeneous porous media of the Corey-Brooks model, [4].

2.1.1. *Basic solutions.* Equations (1)–(2) have solutions that propagate as waves. The Jacobian matrix of the fluxes is the key for rarefaction curves. The characteristic speeds are the two eigenvalues of the Jacobian derivative matrix

$$\mathrm{J}(S) := \frac{\partial(f_\mathrm{w}(S),\, f_\mathrm{g}(S))}{\partial(s_\mathrm{w},\, s_\mathrm{g})} = \frac{\partial F(S)}{\partial S}, \tag{5}$$

provided that these eigenvalues are real, in which case the smaller one is called the slow-family characteristic speed $\lambda_\mathrm{s}(s_\mathrm{w},\, s_\mathrm{g})$ and the larger one is called the fast-family characteristic speed $\lambda_\mathrm{f}(s_\mathrm{w},\, s_\mathrm{g})$. For the Corey model, both eigenvalues are real and nonnegative for each state in the saturation triangle.

The *self-similarity* of solutions of a Riemann problem implies that if $u(x,\,t)$ is such a solution at a given time t, then $u(\alpha x,\, \alpha t)$ is also a solution for any $\alpha > 0$. Centered rarefaction and shock waves are based on self-similarity.

System (1)–(2) has continuous solutions called slow- and fast-family rarefaction waves. They arise by solving an ODE, namely,

$$\{\mathrm{J}(S) - \xi\,\mathrm{I}\}\vec{r}(S) = 0, \qquad \frac{dS}{d\xi} = \vec{r}(S),$$

where $S(\xi)$, for $\xi = x/t$, is the profile of the rarefaction provided ξ is monotonic increasing. Some integral curves appearing in the solution of Riemann problems are plotted in Fig. 1.

This system also admits solutions that have jump discontinuities. The Hugoniot locus of a point S^o, denoted as $\mathcal{H}(S^o)$, is given by all the points S that satisfy the Rankine-Hugoniot (RH) condition:

$$F(S) - F(S^o) = \sigma(S - S^o), \tag{6}$$

where $\sigma = \sigma(S^o,\, S)$ is the velocity of the discontinuity, and the fluxes $F(S)$ and saturations S are given as before. (Notice that S belongs to $\mathcal{H}(S^o)$ if and only if S^o belongs to $\mathcal{H}(S)$.) Admissibility of discontinuites for systems of conservation laws such as (1)–(2) is discussed in [2].

Notice that if the RH condition between states S^a and S^o holds with a certain speed σ, and it also holds for the same speed between states S^o and S^b, it is easy

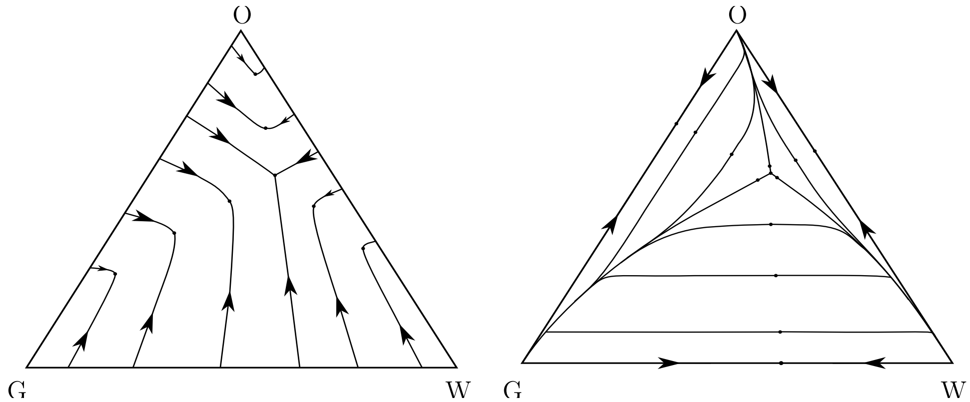

FIGURE 1. Integral curves; slow and fast families. The triple intersection is the umbilic point. Dots on integral curves are inflections, arrows point in the increasing eigenvalue direction. (The specific mobilities are in Remark 2.)

to see that the RH condition is satisfied between states S^a and S^b with the same speed. This is the essence of the triple-shock rule [9]. The definition of σ_{ij} as the shock speed $\sigma(S_i, S_j)$ will be useful.

Theorem 2.2 (Triple-shock rule)**.** *Let the states S_1, S_2 belong to $\mathcal{H}(S_0)$. If $\sigma_{01} = \sigma_{02}$ holds, then S_1 belongs to $\mathcal{H}(S_2)$ and the relations $\sigma_{01} = \sigma_{02} = \sigma_{12}$ hold.*

Proof. Define σ as $\sigma_{01} = \sigma_{02}$. Subtract versions of equation (6) written for (S_0, S_1) and for (S_0, S_2), obtaining $F(S_2) - F(S_1) = \sigma(S_2 - S_1)$, which indicates that S_1 belongs to $\mathcal{H}(S_2)$ and σ_{12} is equal to σ. □

The following variant of Theorem 2.2 has been used in several works appearing in this conference.

Lemma 2.3. *Let S_0, S_1, S_2 be non-collinear states such that S_1, S_2 belong to $\mathcal{H}(S_0)$ and S_1 belongs to $\mathcal{H}(S_2)$. Then $\sigma_{01} = \sigma_{02} = \sigma_{12}$ holds.*

Proof. Let us express the RH relations of the involved states; we have

$$F(S_1) - F(S_0) = \sigma_{01}(S_1 - S_0), \quad F(S_2) - F(S_0) = \sigma_{02}(S_2 - S_0),$$
$$F(S_1) - F(S_2) = \sigma_{12}(S_1 - S_2). \tag{7}$$

By subtracting (7.b) and (7.c) from (7.a), we obtain

$$0 = \sigma_{01}(S_1 - S_0) - \sigma_{02}(S_2 - S_0) - \sigma_{12}(S_1 - S_2).$$

We subtract the trivial relation $0 = \sigma_{12}(S_1 - S_0) - \sigma_{12}(S_2 - S_0) - \sigma_{12}(S_1 - S_2)$ obtaining $0 = (\sigma_{01} - \sigma_{12})(S_1 - S_0) - (\sigma_{02} - \sigma_{12})(S_2 - S_0)$. Recalling that the sates are non-collinear, we notice that the latter relation holds if and only if $\sigma_{01} - \sigma_{12}$ and $\sigma_{02} - \sigma_{12}$ are zero, which proves the lemma. □

A system is called strictly hyperbolic if the characteristic speeds satisfy the inequality $\lambda_{\mathrm{s}}(S) < \lambda_{\mathrm{f}}(S)$ everywhere; they are well studied [10, 11]. In three-phase flow models there are points where the characteristic speeds coincide, which are called coincidence points. Furthermore, in Corey models there are isolated coincidence points where the Jacobian matrix is a multiple of the identity, *i.e.*, umbilic points.

The quadratic Corey model is defined by the permeabilities $k_\alpha(S) = s_\alpha^2$ for α = w, g, o. Such a model is well understood; in particular, the location and characteristics of umbilic points are well known. There is a unique umbilic point $U = (u_{\rm w}, u_{\rm g})$ in the interior of Δ, with $u_{\rm o} = 1 - u_{\rm w} - u_{\rm g}$, the coordinates of which are

$$u_\alpha \;=\; \mu_\alpha/(\mu_{\rm w} + \mu_{\rm g} + \mu_{\rm o}), \qquad \text{for} \qquad \alpha = \text{w, g, o}.$$

Such a point satisfies the following.

Property 2.4. *For the quadratic Corey model, the characteristic speeds are equal to* 2 *at the interior umbilic point.*

Three other umbilic points lie on the vertices of the saturation triangle.

Property 2.5. *For the quadratic Corey model, the shock speed from the interior umbilic point to vertices of the triangle are equal to 1.*

3. **Structures in the saturation triangle for convex Corey models.**

When two of the permeabilities in (4) cease to be scalar multiples of the same convex function, the umbilic point gives rise to two points: the first one is still an umbilic point, and Property 2.4 holds, and at the second one, only Property 2.5 holds. It is because of the shock speed equality that the latter point will be called Equal-Speed Shocks to vertices or ESS.

3.1. **The umbilic point location.**

Inmiscible three-phase flow models are typically non-strictly hyperbolic, except in the model in [7]. Lemma 3.1 follows from results in [14] for the case where the gravity force is not active. (In [8, 15] there are shorter proofs.)

Lemma 3.1. *Consider a convex Corey permeability model, see Definition 2.1. There is always a single point U in the interior of the saturation triangle satisfying*

$$\mathrm{m}_{\rm w}'(u_{\rm w}) = \mathrm{m}_{\rm g}'(u_{\rm g}) = \mathrm{m}_{\rm o}'(u_{\rm o}), \tag{8}$$

which is the unique umbilic point in the interior of the triangle. It has characteristic speed $\lambda(U) = \mathrm{m}_{\rm w}'(u_{\rm w})/\mathrm{m}(U)$.

An important feature of the models considered is the following: from properties (3) and (4) of Definition 2.1, one can see that the Hessian for the total mobility:

$$\begin{pmatrix} \mathrm{m}_{\rm w}'' + \mathrm{m}_{\rm o}'' & \mathrm{m}_{\rm o}'' \\ \mathrm{m}_{\rm o}'' & \mathrm{m}_{\rm g}'' + \mathrm{m}_{\rm o}'' \end{pmatrix} \tag{9}$$

is a positive definite matrix. Hence the motivation of the following result.

Corollary 1. *For a convex Corey type model, the total mobility has a single extremum in the interior of the triangle, which occurs at the umbilic point. The extremum is a minimum.*

Proof. Equating to zero the partial derivatives of m in (3.b) relatively to $s_{\rm w}$ and $s_{\rm g}$ implies $\mathrm{m}_{\rm w}' = \mathrm{m}_{\rm o}'$ as well as $\mathrm{m}_{\rm g}' = \mathrm{m}_{\rm o}'$; then Lemma 3.1 guarantees that this extremum occurs at the single umbilic point. Thus from the positive definiteness of (9) we obtained that this extremum is the minimum. □

Remark 3. Darcy's law says that the total flow rate of a fluid mixture is proportional to the pressure gradient; the proportionality coefficient is (minus) the total mobility. Corollary 1 implies that maximum pressure gradient is needed to displace the fluid mixture at saturations given by the umbilic point, for a given total flow rate. In other words the umbilic point gives the saturation proportion for which each of the three fluids hinders maximally the flow of the other two. (Total flow is minimal for a specific pressure gradient.)

We will call $\mathrm{m}'_\mathrm{w}(s_\mathrm{w})$ the sensitivity of the water mobility to water saturation. The first equality in (8), $\mathrm{m}'_\mathrm{w}(s_\mathrm{w}) = \mathrm{m}'_\mathrm{g}(s_\mathrm{g})$, defines the *equal water-gas sensitivity curve*, which can be parametrized either as a function of s_w or s_g; it contains U and O. Similarly we can define equal water-oil and gas-oil sensitivity curves. See the three dashed curves in Fig. 2. (In the absence of gravitational force these curves were called two-phase-like-flow sets in [14].)

Let us summarize properties of the equal sensitivity curves. First of all, recall that $\mathrm{m}'_\mathrm{w} = \mathrm{m}'_\mathrm{g}$ implies $\partial\mathrm{m}/\partial s_\mathrm{w} = \partial\mathrm{m}/\partial s_\mathrm{g}$, for brevity we call $\partial\mathrm{m}$ such a value, thus the Jacobian matrix at any point of the equal water-gas sensitivity curve is

$$\mathrm{J}(S) = \frac{1}{\mathrm{m}^2}\begin{pmatrix} \mathrm{m}'_\mathrm{w}\mathrm{m} - \mathrm{m}_\mathrm{w}\partial\mathrm{m} & -\mathrm{m}_\mathrm{w}\partial\mathrm{m} \\ -\mathrm{m}_\mathrm{g}\partial\mathrm{m} & \mathrm{m}'_\mathrm{w}\mathrm{m} - \mathrm{m}_\mathrm{g}\partial\mathrm{m} \end{pmatrix}.$$

Along the curve one eigenvalue is $\lambda = \mathrm{m}'_\mathrm{w}/\mathrm{m}$ with eigenvector $(1, -1)$ (in Cartesian coordinates), which is parallel to the side $s_\mathrm{o} = 0$.

Moreover, the total mobility is minimum on the equal sensitivity curve in the direction of such eigenvector. Indeed, $\nabla\mathrm{m}\cdot(1, -1) = \partial\mathrm{m}/\partial s_\mathrm{w} - \partial\mathrm{m}/\partial s_\mathrm{g}$ is zero on the sensitivity curve, which turns out to be at a minimum because the Hessian in (9) is positive definite. (Analogous statements hold for other sensitivity curves.)

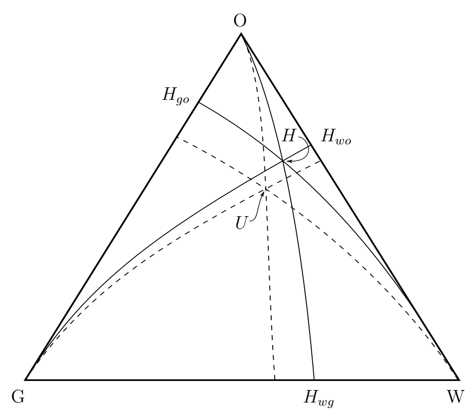

FIGURE 2. Location of umbilic and ESS points. Solid curves are Hugoniot loci from pure saturations. The umbilic location is given from similar dashed curves.

Remark 4. For non-convex Corey models we have the following facts. A converse to Lemma 3.1 holds: an umbilic point in the interior of the triangle satisfies (8). Instead of Corollary 1, every extremum of the total mobility is a coincidence point; such a point is umbilic provided that the second derivatives of two of the mobilities do not vanish simultaneously there. (The extrema are not necessarily unique and do not need to be minima.)

3.2. **The equal-speed shocks curves.** Let us consider the vertex $\mathrm{O} = (0, 0)$, and look for points $S = (s_\mathrm{w}, s_\mathrm{g})$ in Δ satisfying RH relation (6):

$$f_\mathrm{w}(S) = \sigma s_\mathrm{w}, \qquad f_\mathrm{g}(S) = \sigma s_\mathrm{g}; \tag{10}$$

where we used the fact that water and gas saturations for pure oil are zero, water and gas permeabilities are also zero. For the same reason the sides $s_{\rm w} = 0$ and $s_{\rm g} = 0$ are part of the Hugoniot locus of O. A third solution appears equating σ in Eqs. (10) leading to

$$\sigma(S,\,\mathrm{O}) \;=\; \frac{f_{\rm w}(S)}{s_{\rm w}} \;=\; \frac{f_{\rm g}(S)}{s_{\rm g}}; \tag{11}$$

points S satisfying the last equality in (11) form a curve inside Δ.

We denote by $\mathcal{H}_i(\mathrm{O})$ the locus in Δ that satisfies Eq. (11), *i.e.*, the "interior Hugoniot locus" from O; the Hugoniot locus of the vertex O is given by $\mathcal{H}_i(\mathrm{O})$ and the sides WO and GO. Since for any state S on $\mathcal{H}_i(\mathrm{O})$, $\mathcal{H}(S)$ intersects both sides WO and GO, see [5], from Lemma 2.3 we have the following

Claim 3.2. *All points in the internal Hugoniot locus $\mathcal{H}_i(\mathrm{O})$ satisfy the triple-shock rule between* O *and points on the boundary* WO*; they also satisfy the triple-shock rule between* O *and points on the boundary* GO.

We define $\mathcal{H}_i(\mathrm{O})$, from equality (11), as the *equal water-gas shock speed curve* (as we will show presently), which can be parametrized either as a function of $s_{\rm w}$ or $s_{\rm g}$. Actually, since each $\mathrm{m}_\alpha(s_\alpha)$ is an increasing continuous function, its inverse is well defined and increasing. With aid of the constraint $s_{\rm w} + s_{\rm g} + s_{\rm o} = 1$, it is easy to see that points $(s_{\rm w},\, s_{\rm g})$ satisfying the second equality in relation (11) can be parametrized by $s_{\rm o}$, *i.e.*, there exist smooth functions

$$H_{\rm w},\, H_{\rm g} : [0,\,1] \to [0,\,1] \qquad \text{s.t.} \qquad (H_{\rm w}(s_{\rm o}),\, H_{\rm g}(s_{\rm o})) \in \mathcal{H}_i(\mathrm{O}), \tag{12}$$

for all $s_{\rm o} \in [0,\,1]$; notice that $H'_{\rm w}$ and $H'_{\rm g}$ are negative because when $s_{\rm o}$ increases $s_{\rm w} + s_{\rm g}$ decreases. Similarly we can define equal water-oil and gas-oil shock speed curves; $\mathcal{H}_i(\mathrm{G})$ and $\mathcal{H}_i(\mathrm{W})$.

The intersection of $\mathcal{H}_i(\mathrm{W})$, $\mathcal{H}_i(\mathrm{G})$ and $\mathcal{H}_i(\mathrm{O})$, is denoted by $H := (h_{\rm w},\, h_{\rm g})$, with $h_{\rm o} = 1 - h_{\rm w} - h_{\rm g}$, and satisfies

$$\sigma \;=\; \frac{f_{\rm w}(H)}{h_{\rm w}} \;=\; \frac{f_{\rm g}(H)}{h_{\rm g}} \;=\; \frac{f_{\rm o}(H)}{h_{\rm o}}. \tag{13}$$

This is the ESS point (Equal-Speed Shocks); the shock speeds from H to any vertex have the same value σ. Notice from relations (13) that H satisfies $\sigma = \Sigma_\alpha f_\alpha(H)/\Sigma_\alpha h_\alpha = 1$. Defining $H_{\rm wg}$, $H_{\rm wo}$, $H_{\rm go}$ as the intersection of the internal Hugoniot $\mathcal{H}_i(\mathrm{O})$, $\mathcal{H}_i(\mathrm{G})$, $\mathcal{H}_i(\mathrm{W})$ with the sides WG, WO, GO respectively (see Fig. 2), we notice that the triple-shock rule (see Theorem 2.2) holds with speed one for seven points, namely,

$$\sigma(A,\,B) \;=\; 1, \qquad \text{with} \qquad A,\,B \in \{H,\,\mathrm{W},\,\mathrm{G},\,\mathrm{O},\,H_{\rm wg},\,H_{\rm wo},\,H_{\rm go}\},$$

since each point belongs to the Hugoniot locus of the three vertices.

4. **Concluding remark.**

The internal Hugoniot loci of the vertices give rise to the ESS point, while the equal sensitivity curves give rise to the umbilic point.

As in [15] one can follow the ordering of increasing directions of fast rarefaction curves near the boundary, see Fig. 1, and notice that there is an orientation reversal, thus a quadratic expansion of the fluxes about the umbilic point shows that in our case it must be classified as Type I or II, see Fig. 3 and [13].

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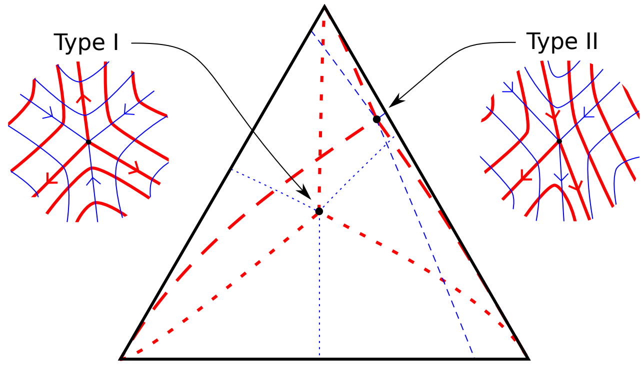

Figure 3. In the saturation triangle there are two possible umbilic point types for Corey permeability models with different viscosities. We represent the two possibilities. The rarefaction behavior around the umbilic type is sketched in the small insets. (Lighter curves represent slow family, darker curves represent fast family.)

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