MATEMÁTICAS APLICADAS A LA ECONOMÍA
CUADERNO DE EJERCICIOS
SOLUCIONES

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INTRODUCCIÓN

Este documento constituye un material de apoyo para el curso de Matemáticas Aplicadas a la Economía, para las carreras de Economía y Dirección Financiera en el ITAM. Contiene las soluciones detalladas del documento de trabajo *Matemáticas Aplicadas a la Economía, Cuaderno de Ejercicios*, Lorena Zogaib, Departamento de Matemáticas, ITAM, enero 2 de 2017.

Todas las soluciones fueron elaboradas por mí, sin una revisión cuidadosa, por lo que seguramente el lector encontrará varios errores en el camino. Ésta es una transcripción en computadora, de mis versiones manuscritas originales. Para este fin, conté con la colaboración de Carlos Gómez Figueroa, que realizó la primera transcripción de las soluciones en Scientific WorkPlace.

Agradezco de antemano sus comentarios y correcciones en relación con este material.

Lorena Zogaib

MATEMÁTICAS APLICADAS A LA ECONOMÍA
TAREA 1 - SOLUCIONES
ECUACIONES EN DIFERENCIAS I
(Temas 1.1-1.3)

1. (a) $x_{t+1} = \sqrt{3t}\,x_t$ es una ecuación no autónoma, lineal, homogénea.
   (b) $x_{t+1}(1 - x_t) = x_t$ es una ecuación autónoma, no lineal.
   (c) $x_t - 3x_{t-1} + 4 = 0$ es una ecuación autónoma, lineal, no homogénea.

2. (a) $x_t = 3^{t+1} + 2 \implies x_{t+1} = 3^{(t+1)+1} + 2 = 3^{t+2} + 2$
   $\therefore 3x_t - 4 = 3(3^{t+1} + 2) - 4 = 3^{t+2} + 6 - 4 = 3^{t+2} + 2 = x_{t+1}$
   $\therefore 3x_t - 4 = x_{t+1}$

   (b) $z_t = t^2 + t \implies z_{t-1} = (t-1)^2 + (t-1)$
   $$\therefore z_t - z_{t-1} = (t^2 + t) - \left[(t-1)^2 + (t-1)\right] = (t^2 + t) - (t^2 - 2t + 1 + t - 1) = 2t$$
   $\therefore z_t - z_{t-1} = 2t$

   (c) $a_t = 2(5)^{t/2} \implies a_{t+1} = 2(5)^{(t+1)/2}$
   $$\therefore a_{t+1}^2 - 5a_t^2 = \left[2(5)^{(t+1)/2}\right]^2 - 5\left[2(5)^{t/2}\right]^2 = 4(5)^{t+1} - 5\left[4(5)^t\right] = 0$$
   $\therefore a_{t+1}^2 = 5a_t^2$

3. La solución de la ecuación lineal autónoma $x_{t+1} = ax_t + b$, con condición inicial x_0, es: i) $x_t = x_0 + bt$, si $a = 1$, ii) $x_t = a^t(x_0 - x^*) + x^*$, si $a \neq 1$, en donde $x^* = \dfrac{b}{1-a}$ es el punto fijo.

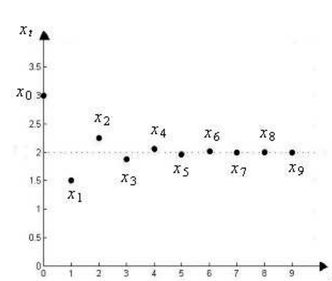
   (a) Ecuación:
   $P_t = 2P_{t-1}$
   Solución:
   El punto fijo es $P^* = 0$. Suponiendo que la población inicial es P_0, la solución es
   $P_t = 2^t P_0, \quad t = 0, 1, 2, \ldots$

   (b) Ecuación:
   $K_{t+1} - K_t = rK_t$, o bien, $K_{t+1} = (1 + r) K_t$
   Solución:
   El punto fijo es $K^* = 0$. Suponiendo que el capital inicial es K_0, la solución es
   $K_t = (1 + r)^t K_0, \quad t = 0, 1, 2, \ldots$

(c) Ecuación:
$K_{t+1} - K_t = rK_0$, o bien, $K_{t+1} = K_t + rK_0$
Solución:
Suponiendo que el capital inicial es K_0, la solución es
$K_t = (1 + rt) K_0, \quad t = 0, 1, 2, \ldots$

(d) Ecuación:
$I_t - I_{t-1} = rI_{t-1} + d$, o bien, $I_t = (1 + r) I_{t-1} + d$
Solución:
El punto fijo es $I^* = -\dfrac{d}{r}$. Suponiendo que la inversión inicial es I_0, la solución es
$I_t = (1 + r)^t \left(I_0 + \dfrac{d}{r} \right) - \dfrac{d}{r}, \quad t = 0, 1, 2, \ldots$

4. (a) $x_{t+1} = -(1/2) x_t + 3$, $x_0 = 3$
Punto fijo:
$x^* = -(1/2) x^* + 3$
$\therefore\ x^* = 2$
Solución:
$x_t = (-1/2)^t (3 - 2) + 2$
$\therefore\ x_t = (-1/2)^t + 2, \quad t = 0, 1, 2, \ldots$
Estabilidad:
$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} \left[(-1/2)^t + 2 \right] = \underbrace{\lim_{t\to\infty} (-1/2)^t}_{=0} + 2 = 2 = x^*$$
$\therefore\ x^* = 2$ es asintóticamente estable.
El sistema presenta convergencia alternante.
Gráfica de la solución:



(b) $2x_{t+1} - 3x_t - 4 = 0$, $x_0 = 0$
Reescribimos la ecuación como $x_{t+1} = \dfrac{3}{2} x_t + 2$

Punto fijo:

$$x^* = \frac{3}{2}x^* + 2$$

$\therefore\ x^* = -4$

Solución:

$$x_t = (3/2)^t\,(0 - (-4)) + (-4)$$

$\therefore\ x_t = 4\,(3/2)^t - 4, \quad t = 0, 1, 2, \ldots$

Estabilidad:

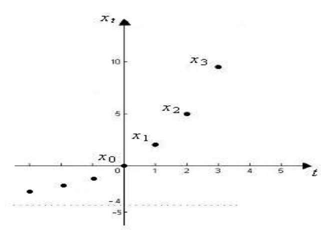
$$\lim_{t\to\infty} x_t = \lim_{t\to\infty}\left[4\,(3/2)^t - 4\right] \text{ diverge}$$

$$\lim_{t\to-\infty} x_t = 4\underbrace{\lim_{t\to-\infty}(3/2)^t}_{=0} - 4 = -4 = x^*$$

$\therefore\ x^* = -4$ es asintóticamente inestable.

El sistema presenta divergencia monótona.

Gráfica de la solución:



(c) $x_{t+1} - x_t = (1/2)\,x_t + 2,\ x_0 = 0$

Este ejercicio es idéntico al del inciso anterior.

(d) $x_{t+1} = -x_t + 5,\ x_0 = 5$

Punto fijo:

$$x^* = -x^* + 5$$

$\therefore\ x^* = \dfrac{5}{2}$

Solución:

$$x_t = (-1)^t\left(5 - \left(\frac{5}{2}\right)\right) + \frac{5}{2}$$

$\therefore\ x_t = \dfrac{5}{2}\left(1 + (-1)^t\right), \quad t = 0, 1, 2, \ldots$

Estabilidad:

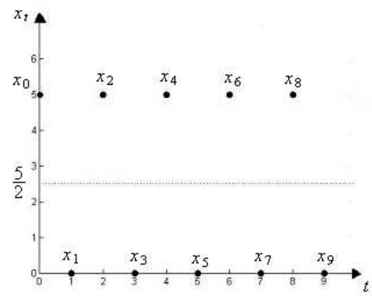
Notamos que

$$x_t = \begin{cases} 5, & \text{si } t \text{ es par} \\ 0, & \text{si } t \text{ es impar} \end{cases}$$

$\therefore$ No existen $\lim\limits_{t\to\infty} x_t$ y $\lim\limits_{t\to-\infty} x_t$.

$\therefore$ El punto fijo no es estable, ni inestable (caso degenerado).

Gráfica de la solución:



(e) $x_{t+1} = -x_t + 5,\ x_0 = \dfrac{5}{2}$

Es la misma ecuación que en (c), pero con diferente x_0.

Punto fijo:

$$x^* = -x^* + 5$$

$\therefore\ x^* = \dfrac{5}{2}$

Solución:

$$x_t = (-1)^t \left(\frac{5}{2} - \left(\frac{5}{2} \right) \right) + \frac{5}{2}$$

$\therefore\ x_t = \dfrac{5}{2}, \quad t = 0, 1, 2, \ldots$

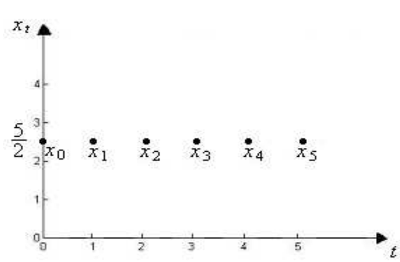
Estabilidad:

La sucesión es constante, con

$$\lim_{t \to \infty} x_t = \lim_{t \to \infty} \frac{5}{2} = \frac{5}{2}$$

$\therefore$ El sistema es estable.

Gráfica de la solución:



(f) $x_{t+1} = x_t + 2,\ x_0 = \dfrac{5}{2}$

Punto fijo:

$$x^* = x^* + 2$$

$\therefore\ 0 = 2 \quad \therefore$ no hay punto fijo

Solución:

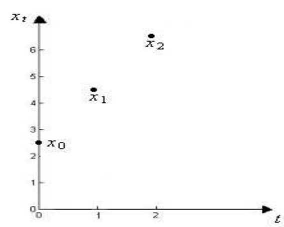
Utilizando el método de iteración, se obtiene

$$x_t = \frac{5}{2} + 2t, \quad t = 0, 1, 2, \ldots$$

Estabilidad:

No hay estabilidad (diverge $\lim\limits_{t \to \pm\infty} x_t$).

Gráfica de la solución:



5. (a) $p_t - p_{t-1} = \beta(\phi - p_t), \quad \phi, \beta > 0$

Precio de equilibrio:

$$p^* - p^* = \beta(\phi - p^*)$$

$\therefore\ p^* = \phi$

(b) Reescribimos la ecuación, como

$$p_t(1+\beta) = p_{t-1} + \beta\phi$$

$$\therefore\ p_t = \left(\frac{1}{1+\beta}\right) p_{t-1} + \frac{\beta\phi}{1+\beta}$$

Solución:

$$p_t = \left(\frac{1}{1+\beta}\right)^t (p_0 - \phi) + \phi, \ \ t = 0, 1, 2, \ldots$$

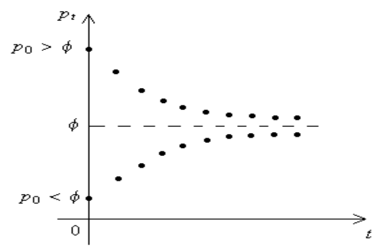
(c) Estabilidad:

Como $0 < \dfrac{1}{1+\beta} < 1$, por lo tanto

$$\lim_{t \to \infty} p_t = (p_0 - \phi) \underbrace{\lim_{t \to \infty} \left(\frac{1}{1+\beta}\right)^t}_{=0} + \phi^* = \phi^*$$

$\therefore$ el precio converge a $p^* = \phi$.

Por último, la convergencia es monótona, ya que $\dfrac{1}{1+\beta} > 0$.



6. $S_t = \alpha Y_t, \quad I_{t+1} = \beta\left(Y_{t+1} - Y_t\right), \quad S_t = I_t, \quad 0 < \alpha < \beta$

$\therefore\ \beta\left(Y_{t+1} - Y_t\right) = I_{t+1} = S_{t+1} = \alpha Y_{t+1}$

$\therefore\ Y_{t+1} = \dfrac{\beta}{\beta - \alpha} Y_t$

La solución es

$$Y_t = \left(\frac{\beta}{\beta - \alpha}\right)^t Y_0, \quad t = 0, 1, 2, \ldots$$

Por último, como $0 < \alpha < \beta$, por lo tanto $\dfrac{\beta}{\beta - \alpha} > 1$. Así,

$\lim\limits_{t\to\infty} Y_t$ diverge,

$$\lim_{t\to-\infty} Y_t = Y_0 \underbrace{\lim_{t\to-\infty}\left(\frac{\beta}{\beta-\alpha}\right)^t}_{=0} = 0.$$

Por lo tanto, el punto fijo $Y^* = 0$ es asintóticamente inestable.

7. $Y_t = C_t + I_t + G_t, \ \ C_t = C_0 + \alpha Y_{t-1}, \quad 0 < \alpha < 1$

$\therefore\ Y_t = \left(C_0 + \alpha Y_{t-1}\right) + I_t + G_t$

$\therefore\ Y_t = \alpha Y_{t-1} + \left(C_0 + I_t + G_t\right)$

Suponiendo que $I_t = I$, $G_t = G$, se tiene

$$Y_t = \alpha Y_{t-1} + C_0 + I + G$$

El punto fijo se obtiene de $Y^* = \alpha Y^* + C_0 + I + G$, de donde

$$Y^* = \frac{C_0 + I + G}{1 - \alpha}$$

En ese caso, la solución a la ecuación para el ingreso es

$$Y_t = \alpha^t\left(Y_0 - Y^*\right) + Y^*, \quad t = 0, 1, 2, \ldots$$

Por último, como $0 < \alpha < 1$, por lo tanto

$$\lim_{t\to\infty} Y_t = (Y_0 - Y^*) \underbrace{\lim_{t\to\infty} \alpha^t}_{=0} + Y^* = Y^*.$$

Así, el punto fijo $Y^* = \dfrac{C_0 + I + G}{1-\alpha}$ es asintóticamente estable.

8. (a) Sea $\beta \neq 1$ y sea

$$S = \sum_{k=0}^{n} \beta^k.$$

Así,

$$\begin{aligned} S &= 1 + \beta + \beta^2 + \cdots + \beta^{n-1} + \beta^n. \\ \beta S &= \beta + \beta^2 + \beta^3 + \cdots + \beta^n + \beta^{n+1} \\ S - \beta S &= 1 - \beta^{n+1} \\ (1-\beta) S &= 1 - \beta^{n+1} \\ S &= \frac{1-\beta^{n+1}}{1-\beta} \\ \sum_{k=0}^{n} \beta^k &= \frac{1-\beta^{n+1}}{1-\beta}. \end{aligned}$$

(b) Como $|\beta| < 1$, se tiene

$$\lim_{n\to\infty} \beta^n = 0.$$

Por lo tanto,

$$\sum_{k=0}^{\infty} \beta^k = \lim_{n\to\infty} \sum_{k=0}^{n} \beta^k = \lim_{n\to\infty} \frac{1-\beta^{n+1}}{1-\beta} = \frac{1}{1-\beta}\left[1 - \lim_{n\to\infty} \beta^{n+1}\right] = \frac{1}{1-\beta}.$$

9. (a) $x_t = t x_{t-1}, \quad x_0 = 1$

$$\begin{aligned} x_0 &= 1 \\ x_1 &= (1)\, x_0 = (1)\,(1) = 1 \\ x_2 &= (2)\, x_1 = (2)\,(1) = 2 \\ x_3 &= (3)\, x_2 = (3)\,(2)\,(1) = 3! \\ x_4 &= (4)\, x_3 = (4)\,(3)\,(2)\,(1) = 4! \\ &\vdots \\ x_t &= t!, \quad t = 0, 1, 2, \ldots \end{aligned}$$

donde se utilizó que $0! = 1$.

(b) $x_t = \dfrac{x_{t-1}}{t+2}$, x_0 dado

$$\begin{aligned}
x_1 &= \frac{1}{3}x_0 \\
x_2 &= \frac{1}{4}x_1 = \frac{1}{4}\left(\frac{1}{3}x_0\right) = \frac{1}{4\cdot 3}x_0 \\
x_3 &= \frac{1}{5}x_2 = \frac{1}{5\cdot 4\cdot 3}x_0 = \frac{2}{5\cdot 4\cdot 3\cdot 2}x_0 = \frac{2!}{5!}x_0 \\
&\vdots \\
x_t &= \frac{2!}{(t+2)!}x_0, \quad t = 0, 1, 2, \ldots
\end{aligned}$$

(c) $x_{t+1} = ax_t + b^t$, x_0 dada, $a, b > 0$

$$\begin{aligned}
x_1 &= ax_0 + b^0 = ax_0 + 1 \\
x_2 &= ax_1 + b^1 = a(ax_0 + 1) + b^1 = a^2x_0 + a + b \\
x_3 &= ax_2 + b^2 = a\left(a^2x_0 + a + b\right) + b^2 = a^3x_0 + a^2 + ab + b^2
\end{aligned}$$

$$\begin{aligned}
x_4 &= ax_3 + b^3 = a\left(a^3x_0 + a^2 + ab + b^2\right) + b^3 = a^4x_0 + a^3 + a^2b + ab^2 + b^3 \\
&\vdots \\
x_t &= a^tx_0 + \sum_{k=0}^{t-1} a^{(t-1)-k}b^k = a^tx_0 + a^{t-1}\sum_{k=0}^{t-1}\left(\frac{b}{a}\right)^k
\end{aligned}$$

Usando el resultado 8a se obtiene

$$\begin{aligned}
x_t &= a^tx_0 + a^{t-1}\,\frac{1-\left(\dfrac{b}{a}\right)^t}{1-\dfrac{b}{a}} \\
&= a^tx_0 + \frac{a^t\left[1-\left(\dfrac{b}{a}\right)^t\right]}{a\left[1-\dfrac{b}{a}\right]} \\
&= a^tx_0 + \frac{a^t - b^t}{a-b} \\
&= a^tx_0 + \frac{b^t - a^t}{b-a} \\
x_t &= \left(x_0 - \frac{1}{b-a}\right)a^t + \frac{1}{b-a}b^t, \quad t = 0, 1, 2, \ldots
\end{aligned}$$

(d) $x_{t+1} = a^t x_t + b$, x_0 dada, $a, b > 0$

$$
\begin{aligned}
x_1 &= a^0 x_0 + b = x_0 + b \\
x_2 &= a^1 x_1 + b = a\left(x_0 + b\right) + b = a x_0 + ab + b \\
x_3 &= a^2 x_2 + b = a^2\left(a x_0 + ab + b\right) + b = \left(a^2 a\right) x_0 + \left(a^2 a\right) b + a^2 b + b \\
x_4 &= a^3 x_3 + b = a^3\left(\left(a^2 a\right) x_0 + \left(a^2 a\right) b + a^2 b + b\right) + b \\
&= \left(a^3 a^2 a\right) x_0 + \left(a^3 a^2 a\right) b + \left(a^3 a^2\right) b + a^3 b + b \\
&= \left(a^3 a^2 a\right) x_0 + \left[\left(a^3 a^2 a\right) + \left(a^3 a^2\right) + a^3 + 1\right] b \\
&= \left(\prod_{s=0}^{3} a^s\right) x_0 + \left[\left(\prod_{s=1}^{3} a^s\right) + \left(\prod_{s=2}^{3} a^s\right) + \left(\prod_{s=3}^{3} a^s\right) + 1\right] b \\
&\vdots \\
x_t &= \left(\prod_{s=0}^{t-1} a^s\right) x_0 + b \sum_{k=0}^{t-1}\left(\prod_{s=k+1}^{t-1} a^s\right), \ t = 0, 1, 2, \ldots
\end{aligned}
$$

donde se usó que el producto $\prod_{s=t}^{t-1} a^s$ de cero términos es 1.

10. (a) $x_{t+1} = \frac{1}{2} x_t + \left(\frac{1}{4}\right)^t$, x_0 dado

$$
\begin{aligned}
x_1 &= \frac{1}{2} x_0 + \left(\frac{1}{4}\right)^0 = \frac{1}{2} x_0 + 1 \\
x_2 &= \frac{1}{2} x_1 + \left(\frac{1}{4}\right)^1 = \frac{1}{2}\left(\frac{1}{2} x_0 + 1\right) + \left(\frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 x_0 + \frac{1}{2} + \left(\frac{1}{4}\right) \\
x_3 &= \frac{1}{2} x_2 + \left(\frac{1}{4}\right)^2 = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 x_0 + \frac{1}{2} + \left(\frac{1}{4}\right)\right) + \left(\frac{1}{4}\right)^2 \\
&= \left(\frac{1}{2}\right)^3 x_0 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 \\
x_4 &= \frac{1}{2} x_3 + \left(\frac{1}{4}\right)^3 = \frac{1}{2}\left(\left(\frac{1}{2}\right)^3 x_0 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2\right) + \left(\frac{1}{4}\right)^3 \\
&= \left(\frac{1}{2}\right)^4 x_0 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 \\
&\vdots \\
x_t &= \left(\frac{1}{2}\right)^t x_0 + \sum_{k=0}^{t-1}\left(\frac{1}{2}\right)^{(t-1)-k}\left(\frac{1}{4}\right)^k = \left(\frac{1}{2}\right)^t x_0 + \left(\frac{1}{2}\right)^{t-1} \sum_{k=0}^{t-1}\left(\frac{1/4}{1/2}\right)^k \\
&= \left(\frac{1}{2}\right)^t x_0 + \left(\frac{1}{2}\right)^{t-1} \sum_{k=0}^{t-1}\left(\frac{1}{2}\right)^k.
\end{aligned}
$$

Usando el resultado 8a se obtiene

$$\begin{aligned} x_t &= \left(\frac{1}{2}\right)^t x_0 + \left(\frac{1}{2}\right)^{t-1} \left(\frac{1-(1/2)^t}{1-(1/2)}\right) \\ &= \left(\frac{1}{2}\right)^t x_0 + 2\left(\frac{1}{2}\right)^{t-1} \left[1-\left(\frac{1}{2}\right)^t\right] \\ &= \left(\frac{1}{2}\right)^t x_0 + 4\left(\frac{1}{2}\right)^{t} \left[1-\left(\frac{1}{2}\right)^t\right]. \end{aligned}$$

Por último, como $\lim_{t\to\infty} (1/2)^t = 0$, se tiene

$$\lim_{t\to\infty} x_t = x_0 \lim_{t\to\infty} \left(\frac{1}{2}\right)^t + 4 \lim_{t\to\infty} \left\{\left(\frac{1}{2}\right)^t \left[1-\left(\frac{1}{2}\right)^t\right]\right\} = 0.$$

Esto mismo se obtendría utilizando la serie 8b.

(b) $x_{t+1} = \frac{1}{2}x_t + \left(\frac{1}{2}\right)^t$, x_0 dado

$$\begin{aligned} x_1 &= \frac{1}{2}x_0 + \left(\frac{1}{2}\right)^0 = \frac{1}{2}x_0 + 1 \\ x_2 &= \frac{1}{2}x_1 + \left(\frac{1}{2}\right)^1 = \frac{1}{2}\left(\frac{1}{2}x_0 + 1\right) + \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 x_0 + 2\left(\frac{1}{2}\right) \\ x_3 &= \frac{1}{2}x_2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 x_0 + \frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^3 x_0 + 3\left(\frac{1}{2}\right)^2 \\ x_4 &= \frac{1}{2}x_3 + \left(\frac{1}{2}\right)^3 = \frac{1}{2}\left(\left(\frac{1}{2}\right)^3 x_0 + 3\left(\frac{1}{2}\right)^2\right) + \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^4 x_0 + 4\left(\frac{1}{2}\right)^3 \\ &\vdots \\ x_t &= \left(\frac{1}{2}\right)^t x_0 + t\left(\frac{1}{2}\right)^{t-1} \end{aligned}$$

Por último, debemos tomar $\lim_{t\to\infty} x_t$. Sabemos que $\lim_{t\to\infty} (1/2)^t = 0$.
Por otra parte, usando la regla de L'Hopital

$$\lim_{t\to\infty} \{t\left(\frac{1}{2}\right)^{t-1}\} = \lim_{t\to\infty} \frac{t}{2^{t-1}} \stackrel{L}{=} \lim_{t\to\infty} \frac{1}{2^{t-1}\ln 2} = 0.$$

De esta manera,

$$\lim_{t\to\infty} x_t = x_0 \lim_{t\to\infty} \left(\frac{1}{2}\right)^t + \lim_{t\to\infty} \{t\left(\frac{1}{2}\right)^{t-1}\} = 0.$$

11. $w_{t+1} = (1+r)\,w_t - c_t, \quad c_t = c_0\gamma^t, \quad w_0$ dada

$$w_{t+1} = (1+r)\,w_t - c_0\gamma^t$$

$$\begin{aligned}
w_1 &= (1+r)\,w_0 - c_0 \\
w_2 &= (1+r)\,w_1 - c_0\gamma = (1+r)\left[(1+r)\,w_0 - c_0\right] - c_0\gamma \\
&= (1+r)^2\,w_0 - (1+r)\,c_0 - c_0\gamma \\
w_3 &= (1+r)\,w_2 - c_0\gamma^2 = (1+r)\left[(1+r)^2\,w_0 - (1+r)\,c_0 - c_0\gamma\right] - c_0\gamma^2 \\
&= (1+r)^3\,w_0 - c_0\left[(1+r)^2 + (1+r)\,\gamma + \gamma^2\right] \\
&\;\vdots \\
w_t &= (1+r)^t\,w_0 - c_0\sum_{k=0}^{t-1}(1+r)^{(t-1)-k}\,\gamma^k \\
&= (1+r)^t\,w_0 - c_0\,(1+r)^{t-1}\sum_{k=0}^{t-1}\left(\frac{\gamma}{1+r}\right)^k \\
&= (1+r)^t\,w_0 - c_0\,(1+r)^{t-1}\,\frac{1-\left(\frac{\gamma}{1+r}\right)^t}{1-\frac{\gamma}{1+r}} \\
&= (1+r)^t\,w_0 - c_0\,\frac{(1+r)^t\left[1-\left(\frac{\gamma}{1+r}\right)^t\right]}{(1+r)\left[1-\frac{\gamma}{1+r}\right]} \\
&= (1+r)^t\,w_0 - c_0\,\frac{(1+r)^t - \gamma^t}{1+r-\gamma} \\
&= \left[w_0 - \frac{c_0}{1+r-\gamma}\right](1+r)^t + \frac{c_0}{1+r-\gamma}\gamma^t, \;\; t = 0, 1, 2, \ldots
\end{aligned}$$

12. (a) $D_t = -3p_t + 10, \quad S_t = p_{t-1} + 2$

$D_t = S_t \Longrightarrow -3p_t + 10 = p_{t-1} + 2$

$\therefore\; p_t = -\frac{1}{3}p_{t-1} + \frac{8}{3}$

Punto fijo:

$$p^* = -\frac{1}{3}p^* + \frac{8}{3} \qquad \therefore \qquad p^* = 2$$

Solución:

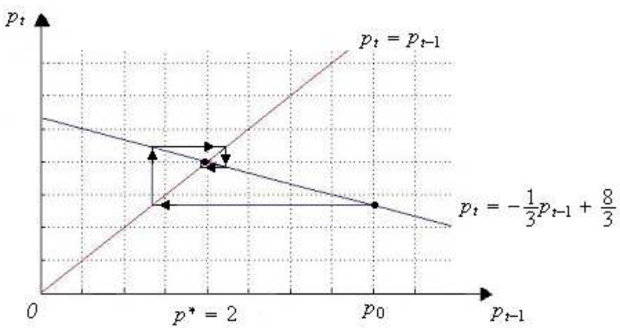
$$p_t = \left(-\frac{1}{3}\right)^t (p_0 - 2) + 2$$

Estabilidad:

$$\lim_{t\to\infty} p_t = (p_0 - 2)\underbrace{\lim_{t\to\infty}\left(-\frac{1}{3}\right)^t}_{=0} + 2 = 2 = p^*$$

$\therefore\; p^* = 2$ es asintóticamente estable.

Diagrama de fase:



(b) $D_t = -4p_t + 25, \quad S_t = 4p_{t-1} + 3$

$D_t = S_t \Longrightarrow -4p_t + 25 = 4p_{t-1} + 3$

$\therefore\ p_t = -p_{t-1} + \dfrac{11}{2}$

Punto fijo:

$$p^* = -p^* + \frac{11}{2}$$

$\therefore\ p^* = \dfrac{11}{4}$

Solución:

$$p_t = (-1)^t \left(p_0 - \frac{11}{4}\right) + \frac{11}{4}$$

Estabilidad:

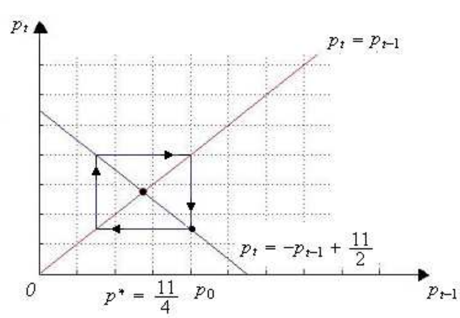
Notamos que

$$p_t = \begin{cases} p_0, & \text{si } t \text{ es par} \\ \dfrac{11}{2} - p_0, & \text{si } t \text{ es impar.} \end{cases}$$

$\therefore$ No existen $\lim\limits_{t\to\infty} p_t$ y $\lim\limits_{t\to-\infty} p_t$ (hay dos puntos de acumulación)

$\therefore$ El sistema no es estable, ni inestable. Se trata de un caso degenerado.

Diagrama de fase:



(c) $D_t = -\left(5/2\right) p_t + 45, \quad S_t = \left(15/2\right) p_{t-1} + 5$

$D_t = S_t \Longrightarrow -\left(5/2\right) p_t + 45 = \left(15/2\right) p_{t-1} + 5$

$\therefore\ p_t = -3p_{t-1} + 16$

Punto fijo:

$$p^* = -3p^* + 16$$

$\therefore\ p^* = 4$

Solución:

$$p_t = \left(-3\right)^t \left(p_0 - 4\right) + 4$$

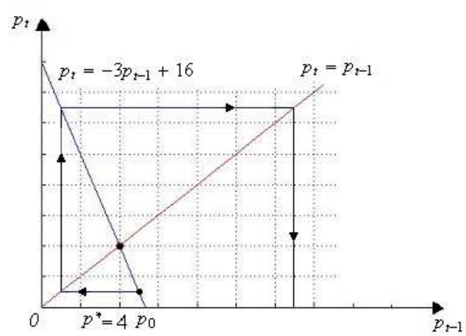
Estabilidad:

$$\lim_{t\to\infty} p_t = \lim_{t\to\infty} \left[\left(-3\right)^t \left(p_0 - 4\right) + 4\right] \text{ diverge}$$

$$\lim_{t\to-\infty} p_t = \left(p_0 - 4\right) \underbrace{\lim_{t\to-\infty} \left(-3\right)^t}_{=0} + 4 = 4 = p^*$$

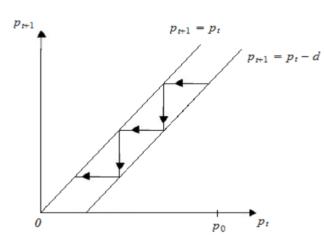
$\therefore\ p^* = 4$ es asintóticamente inestable.

Diagrama de fase:



13. $p_{t+1} = p_t - d, \quad d > 0.$

(a)



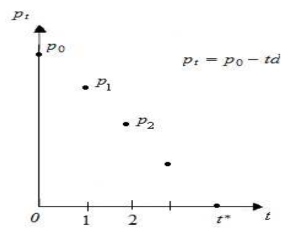
(b) $p_1 = p_0 - d$

$p_2 = p_1 - d = \left(p_0 - d\right) - d = p_0 - 2d$

$\vdots$

$p_t = p_0 - td, \quad t = 0, 1, 2, \ldots$

El precio del bien es igual a cero cuando $t^* = \dfrac{p_0}{d}$.



14. (a) $x_{t+1} = \dfrac{kx_t}{b + x_t}, \quad k, b > 0$

Puntos fijos:

$x^* = \dfrac{kx^*}{b + x^*}$

$x^* (b + x^*) = kx^*$

$x^* (x^* + b - k) = 0$

Un punto fijo es $x_1^* = 0$. El otro punto fijo, $x_2^* = k - b$, sólo tiene sentido si $b < k$, ya que la población x^* es no negativa.

(b) Estabilidad:

Sea $f(x) = \dfrac{kx}{b + x}$

$\therefore\ f'(x) = \dfrac{kb}{(b + x)^2}$

$\therefore\ f'(0) = \dfrac{k}{b}$ y $f'(k - b) = \dfrac{b}{k}$.

Suponiendo que $b < k$, con $k, b > 0$, se tiene

$\therefore\ f'(0) > 1$ y $f'(k - b) < 1$

$\therefore\ x_1^* = 0$ es asintóticamente inestable y $x_2^* = k - b$ es asintóticamente estable.

15. $y_{t+1} (a + by_t) = cy_t, \quad y_{t+1} (a + by_t) = cy_t, \ \ a, b, c > 0$ y $y_0 > 0$

(a) Partimos de $y_{t+1} = \dfrac{cy_t}{a + by_t}$, con $a, b, c > 0$. Como $y_0 > 0$, por lo tanto $y_1 > 0$. Con este mismo razonamiento, se sigue que $y_2 > 0$, etc... De esta manera, $y_t > 0$ para todo t.

(b) Sea $x_t = 1/y_t$. Sustituyendo $y_t = 1/x_t$ en la ecuación $y_{t+1} = \dfrac{cy_t}{a + by_t}$ se tiene

$$\frac{1}{x_{t+1}} = \frac{c\left(\dfrac{1}{x_t}\right)}{a + b\left(\dfrac{1}{x_t}\right)} = \frac{c}{ax_t + b}$$

$\therefore\ x_{t+1} = \frac{a}{c}x_t + \frac{b}{c}$ $\longleftarrow$ecuación lineal para x_t

En particular, para $y_{t+1}(2+3y_t) = 4y_t,\ y_0 = 1/2$, se obtiene

$$x_{t+1} = \frac{1}{2}x_t + \frac{3}{4},\ \ x_0 = \frac{1}{y_0} = 2$$

$$\therefore\ x_t = \frac{1}{2}\left(\frac{1}{2}\right)^t + \frac{3}{2} = \left(\frac{1}{2}\right)^{t+1} + \frac{3}{2}$$

$$\therefore\ y_t = \frac{1}{\left(\frac{1}{2}\right)^{t+1} + \frac{3}{2}} \qquad t = 0, 1, 2, \ldots$$

Por último,

$$\lim_{t\to\infty} y_t = \frac{1}{0 + \frac{3}{2}} = \frac{2}{3}$$

16. (a) $x_{t+1} = \sqrt{4x_t - 3},\ \ x_t \geq \frac{3}{4}$

Puntos fijos:

$$x^* = \sqrt{4x^* - 3}$$
$$(x^*)^2 = 4x^* - 3$$
$$(x^*)^2 - 4x^* + 3 = 0$$
$$(x^* - 1)(x^* - 3) = 0$$
$$x_1^* = 1,\ \ x_2^* = 3$$

Estabilidad:

Sea $f(x) = \sqrt{4x - 3}$

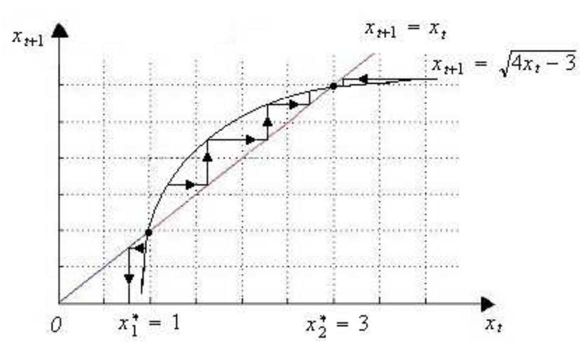
$\therefore\ f'(x) = \frac{2}{\sqrt{4x-3}}$

$\therefore\ f'(1) = 2$ y $f'(3) = \frac{2}{3}$

$\therefore\ |f'(1)| > 1$ y $|f'(3)| < 1$

$\therefore\ x_1^* = 1$ es asintóticamente inestable y $x_2^* = 3$ es asintóticamente estable.

Diagrama de fase:



Para $x_0 \neq 1, 3$ se tiene:

$\frac{3}{4} \leq x_0 < 1 \implies \lim_{t \to -\infty} x_t = 1,$

$1 < x_0 < 3 \implies \lim_{t \to -\infty} x_t = 1 \quad \text{y} \quad \lim_{t \to \infty} x_t = 3,$

$x_0 > 3 \implies \lim_{t \to \infty} x_t = 3.$

(b) $x_{t+1} = x_t^3$

Puntos fijos:

$x^* = (x^*)^3$

$\therefore\ x^* - (x^*)^3 = 0$

$\therefore\ x^* \left(1 - (x^*)^2\right) = 0$

$\therefore\ x^* (1 + x^*)(1 - x^*) = 0$

$\therefore\ x_1^* = -1,\ \ x_2^* = 0,\ \ x_3^* = 1$

Estabilidad:

Sea $f(x) = x^3$

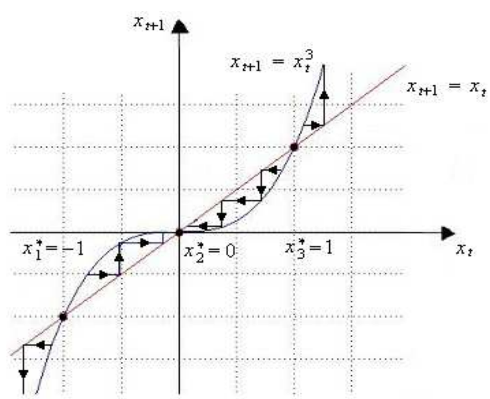
$\therefore\ f'(x) = 3x^2$

$\therefore\ f'(0) = 0,\ f'(-1) = f'(1) = 3$

$\therefore\ |f'(0)| < 1 \ \text{ y } \ |f'(-1)| = |f'(1)| > 1$

$\therefore$ $x_2^* = 0$ es asintóticamente estable; $x_1^* = -1$ y $x_3^* = 1$ son asintóticamente inestables.

Diagrama de fase:



Para $x_0 \neq -1, 0, 1$ se tiene:

$x_0 < -1 \implies \lim_{t \to -\infty} x_t = -1,$

$-1 < x_0 < 0 \implies \lim_{t \to -\infty} x_t = -1 \quad \text{y} \quad \lim_{t \to \infty} x_t = 0,$

$0 < x_0 < 1 \implies \lim_{t \to -\infty} x_t = 1 \quad \text{y} \quad \lim_{t \to \infty} x_t = 0,$

$x_0 > 1 \implies \lim_{t \to -\infty} x_t = 1.$

(c) $x_{t+1} = \dfrac{1}{x_t^2}, \quad x_t > 0$

Puntos fijos:

$$x^* = \frac{1}{(x^*)^2}$$

$\therefore\ (x^*)^3 = 1$

$\therefore\ (x^*)^3 - 1 = 0$

$\therefore\ (x^* - 1)\underbrace{\left((x^*)^2 + x^* + 1\right)}_{\neq 0} = 0$

$\therefore\ x^* = 1$

Estabilidad:

Sea $f(x) = \dfrac{1}{x^2}$

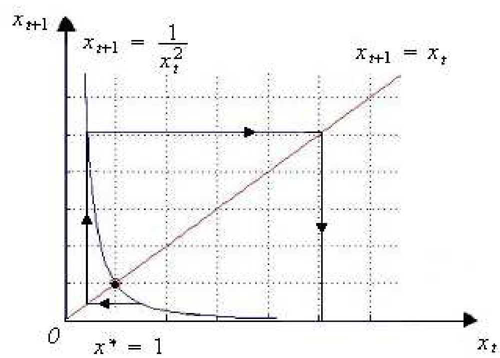
$\therefore\ f'(x) = -\dfrac{2}{x^3}$

$\therefore\ f'(1) = -2$

$\therefore\ |f'(1)| > 1$

$\therefore\ x^* = 1$ es asintóticamente inestable.

Diagrama de fase:



Para $x_0 \neq 1$ se tiene:

$$\lim_{t \to -\infty} x_t = 1.$$

(d) $x_{t+1} = \dfrac{1}{x_t}, \qquad x_t > 0$

Puntos fijos:

$$x^* = \frac{1}{x^*}$$

$(x^*)^2 = 1$

$\therefore\ x^* = 1 \quad (x^* = -1$ se descarta, ya que $x_t > 0)$

Estabilidad:

Sea $f(x) = \frac{1}{x}$

$\therefore\ f'(x) = -\frac{1}{x^2}$

$\therefore\ f'(1) = -1$

$\therefore\ |f'(1)| = 1$

$\therefore$ no se puede aplicar el teorema.

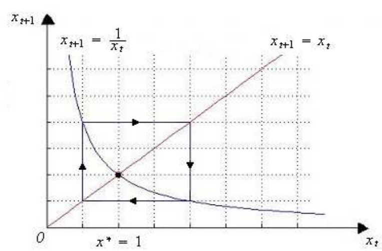
Observamos que

$$x_t = \begin{cases} x_0, & \text{si } t \text{ es par} \\ \frac{1}{x_0}, & \text{si } t \text{ es impar.} \end{cases}$$

$\therefore$ hay dos puntos de acumulación (caso degenerado)

$\therefore$ el sistema no es estable, ni inestable.

Diagrama de fase:



Para $x_0 \neq 1$ se tiene:

$\lim_{t\to\infty} x_t$ y $\lim_{t\to-\infty} x_t$ divergen ambos.

(e) $x_{t+1} = x_t + x_t^3$

Puntos fijos:

$x^* = x^* + (x^*)^3$

$\therefore\ x^* = 0$

Estabilidad:

Sea $f(x) = x + x^3$

$\therefore\ f'(x) = 1 + 3x^2$

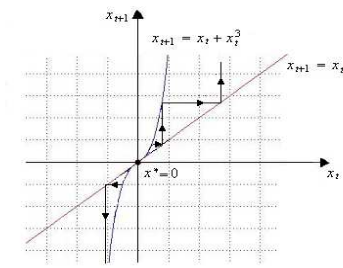
$\therefore\ f'(0) = 1$

$\therefore\ |f'(0)| = 1$

$\therefore$ no se puede aplicar el teorema.

Observamos que $f'(x) > 1$ alrededor de $x = 0$, por lo que $x^* = 0$ es asintóticamente inestable.

Diagrama de fase:



Para $x_0 \neq 0$ se tiene:

$$\lim_{t \to -\infty} x_t = 0.$$

(f) $x_{t+1} = e^{(x_t)-1}$. Nota: el punto fijo es $x^* = 1$

Puntos fijos:

$$x^* = e^{x^*-1}$$

$\therefore$ $x^* = 1$

Estabilidad:

Sea $f(x) = e^{x-1}$

$\therefore$ $f'(x) = e^{x-1}$

$\therefore$ $f'(1) = 1$

$\therefore$ $|f'(1)| = 1$

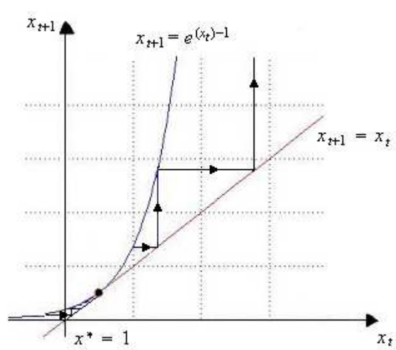
$\therefore$ no se puede aplicar el teorema.

Observamos que $f'(x) > 1$ para $x > 1$, y $f'(x) < 1$ para $x < 1$. Por lo tanto,

$x_0 \leq 1 \Longrightarrow x^* = 1$ es asintóticamente estable,

$x_0 > 1 \Longrightarrow x^* = 1$ es asintóticamente inestable.

Diagrama de fase:



Para $x_0 \neq 1$ se tiene:

$x_0 < 1 \implies \lim_{t\to\infty} x_t = 1$

$x_0 > 1 \implies \lim_{t\to-\infty} x_t = 1$

17. (a) $x_{t+1} = \sqrt{4x_t - 3}, \ x_t \geq \frac{3}{4}$

La ecuación no posee una solución simple.

(b) $x_{t+1} = x_t^3$

Iterando, se obtiene $x_t = (x_0)^{3^t}$.

(c) $x_{t+1} = \frac{1}{x_t^2}, \quad x_t > 0$

Iterando, se obtiene $x_t = (x_0)^{(-2)^t}$

(d) $x_{t+1} = \frac{1}{x_t}, \quad x_t > 0$

Iterando, se obtiene $x_t = (x_0)^{(-1)^t} = \begin{cases} x_0, & \text{si } t \text{ es par} \\ \frac{1}{x_0}, & \text{si } t \text{ es impar.} \end{cases}$

(solución cíclica)

(e) $x_{t+1} = x_t + x_t^3$

La ecuación no posee una solución simple.

(f) $x_{t+1} = e^{(x_t)-1}$

La ecuación no posee una solución simple.

18. (a) $x_{t+1} = x_t^2 + c$

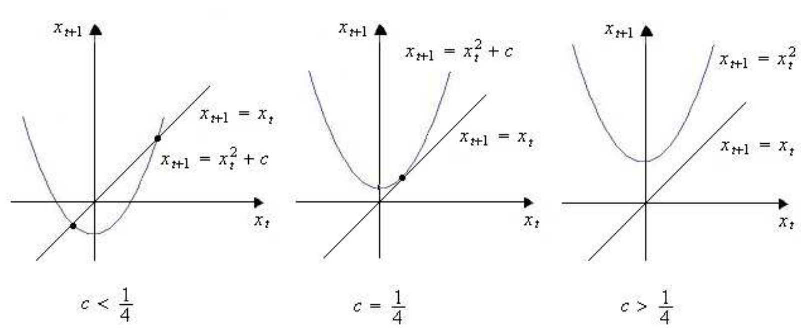
Puntos fijos:

$x^* = (x^*)^2 + c$

$(x^*)^2 - x^* + c = 0$

$\therefore \ x^* = \frac{1 \pm \sqrt{1-4c}}{2}$

De aquí se siguen tres casos: si $c < \frac{1}{4}$ hay dos puntos fijos, si $c = \frac{1}{4}$ sólo hay un punto fijo, y si $c > \frac{1}{4}$ no hay puntos fijos.



(b) $x_{t+1} = x_t^2 - 2, \quad x_0 = \sqrt{2}$

Puntos fijos:

$x^* = (x^*)^2 - 2$

$(x^*)^2 - x^* - 2 = 0$

$(x^* + 1)(x^* - 2) = 0$

$x_1^* = -1, \ \ x_2^* = 2$

Obtenemos la sucesión de puntos a partir del punto inicial dado:

$x_0 = \sqrt{2}$

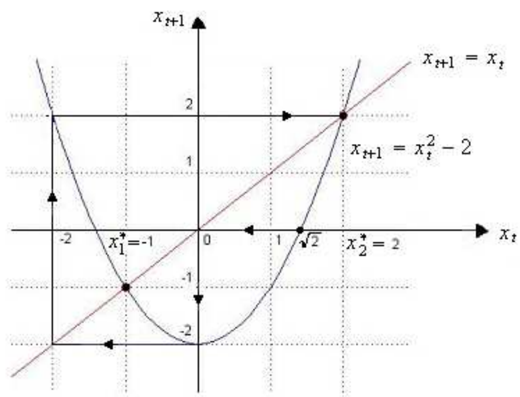
$x_1 = \left(\sqrt{2}.\right)^2 - 2 = 0$

$x_2 = 0^2 - 2 = -2$

$x_3 = (-2)^2 - 2 = 2$

$x_4 = (2)^2 - 2 = 2$

$\therefore \text{orb}\left(\sqrt{2}\right) = \left\{\sqrt{2}, 0, -2, 2, 2, 2...\right\}$



Nota: Aquí no se aplica el teorema sobre el valor de $|f'(x^*)|$, ya que aquí la convergencia no es asintótica, sino que ocurre en el período 3. De hecho, $x_2^* = 2$ es estable para $x_0 = \sqrt{2}$, aunque $|f'(2)| = 4$ es mayor que 1.

MATEMÁTICAS APLICADAS A LA ECONOMÍA
TAREA 2 - SOLUCIONES
ECUACIONES EN DIFERENCIAS II
(Tema 2.1)

1. (a) $x_{t+2} - 3x_{t+1} + 2x_t = 0; \quad x_t = A + B\,2^t$

   $x_t = A + B\,2^t \implies x_{t+1} = A + B\,2^{t+1} \implies x_{t+2} = A + B\,2^{t+2}$

   $$\begin{aligned} \therefore\; x_{t+2} - 3x_{t+1} + 2x_t &= (A + B\,2^{t+2}) - 3\,(A + B\,2^{t+1}) + 2\,(A + B\,2^t) \\ &= A\,(1 - 3 + 2) + B\,(2^{t+2} - 3\,2^{t+1} + 2\,2^t) \\ &= B\,2^t\,(2^2 - 3\,(2) + 2) = 0 \end{aligned}$$

   (b) $x_{t+2} - 2x_{t+1} + x_t = 0; \quad x_t = A + Bt$

   $x_t = A + Bt \Longrightarrow x_{t+1} = A + B\,(t+1) \Longrightarrow x_{t+2} = A + B\,(t+2)$

   $$\begin{aligned} \therefore\; x_{t+2} - 2x_{t+1} + x_t &= (A + Bt + 2B) - 2\,(A + Bt + B) + (A + Bt) \\ &= A\,(1 - 2 + 1) + B\,(2 - 2) + Bt\,(1 - 2 + 1) = 0 \end{aligned}$$

2. (a) $x_{t+2} - 5x_{t+1} + 6x_t = 0$

   Solución:
   Proponemos $x_t = \lambda^t$
   $\therefore\; \lambda^{t+2} - 5\lambda^{t+1} + 6\lambda^t = 0$
   $\therefore\; \lambda^t \left(\lambda^2 - 5\lambda + 6\right) = 0$
   $\therefore\; \lambda^2 - 5\lambda + 6 = 0$
   $\therefore\; (\lambda - 2)\,(\lambda - 3) = 0$
   $\therefore\; \lambda_1 = 2,\;\; \lambda_2 = 3 \qquad$ (raíces reales distintas)
   $\therefore\; x_t = k_1\,2^t + k_2\,3^t, \quad t = 0, 1, 2, \ldots$

   Estabilidad:
   $$\lim_{t\to\infty} x_t = \lim_{t\to\infty} [k_1\,2^t + k_2\,3^t] \text{ diverge}$$
   $$\lim_{t\to-\infty} x_t = \lim_{t\to-\infty} [k_1\,2^t + k_2\,3^t] = 0$$
   $\therefore$El sistema es asintóticamente inestable.

   (b) $x_{t+2} - x_t = 0$

   Solución:
   Proponemos $x_t = \lambda^t$
   $\therefore\; \lambda^2 - 1 = 0$
   $\therefore\; (\lambda + 1)\,(\lambda - 1) = 0$
   $\therefore\; \lambda_1 = -1,\;\; \lambda_2 = 1 \qquad$ (raíces reales distintas)
   $\therefore\; x_t = k_1\,(-1)^t + k_2, \quad t = 0, 1, 2, \ldots$

Estabilidad:

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} \left[k_1 (-1)^t + k_2\right] \text{ diverge}$$

$$\lim_{t\to-\infty} x_t = \lim_{t\to-\infty} \left[k_1 (-1)^t + k_2\right] \text{ diverge}$$

$\therefore$ El sistema no es estable ni inestable (caso degenerado).

(c) $x_{t+2} - 2x_{t+1} + 4x_t = 0$

Solución:

Proponemos $x_t = \lambda^t$

$\therefore\ \lambda^2 - 2\lambda + 4 = 0$

$$\therefore\ \lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4(4)}}{2} = \frac{2 \pm 2\sqrt{-3}}{2} = 1 \pm \left(\sqrt{3}\right) i$$

$\therefore\ \lambda_1 = 1 + \sqrt{3}i,\quad \lambda_2 = 1 - \sqrt{3}i$ (raíces complejas)

$$\therefore \alpha = 1,\ \beta = \sqrt{3},\ \|\lambda\| = \sqrt{\alpha^2 + \beta^2} = 2,\ \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \frac{\pi}{3}$$

$$\therefore\ x_t = 2^t \left[k_1 \cos\left(\frac{\pi}{3}t\right) + k_2\, sen\left(\frac{\pi}{3}t\right)\right], \quad t = 0, 1, 2, \ldots$$

Estabilidad:

Nota que $|x_t| < 2^t (k_1 + k_2)$. Por lo tanto,

$$\lim_{t\to\infty} x_t \text{ diverge}$$

$$\lim_{t\to-\infty} x_t = 0 \quad \text{(usando el teorema del sandwich)}$$

$\therefore$ El sistema es asintóticamente inestable.

(d) $9x_{t+2} - 6x_{t+1} + x_t = 0,\ x_0 = 1,\ x_1 = 2$

Solución:

Proponemos $x_t = \lambda^t$

$\therefore\ 9\lambda^2 - 6\lambda + 1 = 0$

$\therefore\ (3\lambda - 1)^2 = 0$

$\therefore\ \lambda_1 = \lambda_2 = \frac{1}{3}$ (raíces reales repetidas)

$$\therefore\ x_t = k_1 \left(\frac{1}{3}\right)^t + k_2\, t \left(\frac{1}{3}\right)^t$$

Condiciones iniciales:

$$x_0 = 1 = k_1$$

$$x_1 = 2 = k_1 \left(\frac{1}{3}\right) + k_2 \left(\frac{1}{3}\right)$$

$\therefore\ k_1 = 1,\quad k_2 = 5$

$$\therefore\ x_t = \left(\frac{1}{3}\right)^t + 5\,t\left(\frac{1}{3}\right)^t = (1+5t)\left(\frac{1}{3}\right)^t, \quad t = 0, 1, 2, \ldots$$

Estabilidad:

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty}\frac{1+5t}{3^t} \overset{L}{=} \lim_{t\to\infty}\frac{5}{3^t\ln 3} = 0 \text{ (Regla de L'Hopital)}$$

$$\lim_{t\to-\infty} x_t \quad \text{diverge}$$

$\therefore$ El sistema es asintóticamente estable.

3. La solución de la ecuación no homogénea $ax_{t+2}+bx_{t+1}+cx_t = d_t$ está dada por $x_t = x_t^{(h)} + x_t^{(p)}$, donde $x_t^{(h)}$ es la solución general de la ecuación homogénea asociada y $x_t^{(p)}$ es cualquier solución particular de la ecuación no homogénea.

   (a) $x_{t+2} + \frac{1}{4}x_t = 5$

   $x_t^{(h)}: \quad x_{t+2}^{(h)} + \frac{1}{4}x_t^{(h)} = 0$

   Proponemos $x_t^{(h)} = \lambda^t$

   $$\therefore\ \lambda^2 + \frac{1}{4} = 0$$

   $$\therefore\ \lambda_{1,2} = \frac{0 \pm \sqrt{-1}}{2} = \pm\frac{1}{2}i$$

   $$\therefore\ \lambda_1 = \frac{1}{2}i, \quad \lambda_2 = -\frac{1}{2}i \quad \text{(raíces complejas)}$$

   $$\therefore\ \alpha = 0, \quad \beta = \frac{1}{2}, \quad \|\lambda\| = \sqrt{\alpha^2+\beta^2} = \frac{1}{2}$$

   $$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

   $$\therefore\ x_t^{(h)} = \left(\frac{1}{2}\right)^t\left[k_1\cos\left(\frac{\pi}{2}t\right) + k_2\,sen\left(\frac{\pi}{2}t\right)\right]$$

   $x_t^{(p)}: \quad x_{t+2}^{(p)} + \frac{1}{4}x_t^{(p)} = 5$

   Proponemos $x_t^{(p)} = A$

   $$\therefore\ x_{t+1}^{p} = x_{t+2}^{p} = A$$

   $$\therefore\ A + \frac{1}{4}A = 5$$

   $$\therefore\ A = 4$$

   $$\therefore\ x_t^{(p)} = 4$$

   $$\therefore\ x_t = \left(\frac{1}{2}\right)^t\left[k_1\cos\left(\frac{\pi}{2}t\right) + k_2\,sen\left(\frac{\pi}{2}t\right)\right] + 4, \quad t = 0, 1, 2, \ldots$$

(b) $x_{t+2} - 4x_t = 3,\ x_0 = 2,\ x_1 = 1$

$x_t^{(h)}:\quad x_{t+2}^{(h)} - 4x_t^{(h)} = 0$

Proponemos $x_t^{(h)} = \lambda^t$

$\therefore\ \lambda^2 - 4 = 0$

$\therefore\ (\lambda + 2)(\lambda - 2) = 0$

$\therefore\ \lambda_1 = -2, \quad \lambda_2 = 2$

$\therefore\ x_t^{(h)} = k_1 (-2)^t + k_2\, 2^t$

$x_t^{(p)}:\quad x_{t+2}^{(p)} - 4x_t^{(p)} = 3$

Proponemos $x_t^{(p)} = A$

$\therefore\ x_{t+1}^p = x_{t+2}^p = A$

$\therefore\ A - 4A = 3$

$\therefore\ A = -1$

$\therefore\ x_t^{(p)} = -1$

$\therefore\ x_t = k_1 (-2)^t + k_2\, 2^t - 1$

Condiciones iniciales:

$x_0 = 2 = k_1 + k_2 - 1$

$x_1 = 1 = -2k_1 + 2k_2 - 1$

$\therefore\ k_1 = 1,\ \ k_2 = 2$

$\therefore\ x_t = (-2)^t + 2\,(2^t) - 1 = (-2)^t + 2^{t+1} - 1, \quad t = 0, 1, 2, \ldots$

(c) $x_{t+2} - 4x_t = -9t,\ x_0 = 2,\ x_1 = 1$

$x_t^{(h)}:\quad x_{t+2}^{(h)} - 4x_t^{(h)} = 0$

$\therefore\ x_t^{(h)} = k_1 (-2)^t + k_2\, 2^t$ (ver inciso anterior)

$x_t^{(p)}:\quad x_{t+2}^{(p)} - 4x_t^{(p)} = -9t$

Proponemos $x_t^{(p)} = At + B$

$\therefore\ x_{t+1}^p = A(t+1) + B$

$\therefore\ x_{t+2}^p = A(t+2) + B$

$\therefore\ [A(t+2) + B] - 4[At + B] = -9t$

$\therefore\ -3At + (2A - 3B) = -9t$

$\therefore\ -3A = -9$ y $2A - 3B = 0$

$\therefore\ A = 3$ y $B = 2$

$\therefore\ x_t^{(p)} = 3t + 2$

$\therefore\ x_t = k_1 (-2)^t + k_2\, 2^t + 3t + 2$

Condiciones iniciales:

$x_0 = 2 = k_1 + k_2 + 2$

$x_1 = 1 = -2k_1 + 2k_2 + 3 + 2$

$\therefore\ k_1 = 1,\ \ k_2 = -1$

$\therefore\ x_t = (-2)^t - 2^t + 3t + 2, \quad t = 0, 1, 2, \ldots$

(d) $x_{t+2} - 7x_{t+1} + 12x_t = 5\left(2^t\right),\ x_0 = \dfrac{1}{2},\ x_1 = 0$

$x_t^{(h)}: \quad x_{t+2}^{(h)} - 7x_{t+1}^{(h)} + 12x_t^{(h)} = 0$

Proponemos $x_t^{(h)} = \lambda^t$

$\therefore\ \lambda^2 - 7\lambda + 12 = 0$

$\therefore\ (\lambda - 3)(\lambda - 4) = 0$

$\therefore\ \lambda_1 = 3, \quad \lambda_2 = 4$

$\therefore\ x_t^{(h)} = k_1\, 3^t + k_2\, 4^t$

$x_t^{(p)}: \quad x_{t+2}^{(p)} - 7x_{t+1}^{(p)} + 12x_t^{(p)} = 5\left(2^t\right)$

Proponemos $x_t^{(p)} = A\, 2^t$

$\therefore\ x_{t+1}^p = A\, 2^{t+1} = 2A\, 2^t$

$\therefore\ x_{t+2}^p = A\, 2^{t+2} = 4A\, 2^t$

$\therefore\ \left[4A\, 2^t\right] - 7\left[2A\, 2^t\right] + 12\left[A\, 2^t\right] = 5\left(2^t\right)$

$\therefore\ A = \dfrac{5}{2}$

$\therefore\ x_t^{(p)} = \dfrac{5}{2}\left(2^t\right)$

$\therefore\ x_t = k_1\, 3^t + k_2\, 4^t + \dfrac{5}{2}\left(2^t\right)$

Condiciones iniciales:

$x_0 = \dfrac{1}{2} = k_1 + k_2 + \dfrac{5}{2}$

$x_1 = 0 = 3k_1 + 4k_2 + 5$

$\therefore\ k_1 = -3,\ \ k_2 = 1$

$\therefore\ x_t = -3^{t+1} + 4^t + \dfrac{5}{2}\left(2^t\right), \quad t = 0, 1, 2, \ldots$

(e) $x_{t+2} - 3x_{t+1} + 2x_t = 3\left(5^t\right)$

$x_t^{(h)}: \quad x_{t+2}^{(h)} - 3x_{t+1}^{(h)} + 2x_t^{(h)} = 0$

Proponemos $x_t^{(h)} = \lambda^t$

$\therefore\ \lambda^2 - 3\lambda + 2 = 0$

$$\therefore\ (\lambda-1)(\lambda-2)=0$$
$$\therefore\ \lambda_1=1,\quad \lambda_2=2$$
$$\therefore\ x_t^{(h)}=k_1+k_2\,2^t$$

$x_t^{(p)}:\quad x_{t+2}^{(p)}-3x_{t+1}^{(p)}+2x_t^{(p)}=3\,(5^t)$

Proponemos $x_t^{(p)}=A5^t$

$$\therefore\ x_{t+1}^{p}=A5^{t+1}=5\,(A5^t)$$
$$\therefore\ x_{t+2}^{p}=A5^{t+2}=25\,(A5^t)$$
$$\therefore\ [25A5^t]-3\,[5A5^t]+2\,[A5^t]=3\,(5^t)$$
$$\therefore\ 25A-15A+2A=3$$
$$\therefore\ A=\frac{1}{4}$$
$$\therefore\ x_t^{(p)}=\frac{1}{4}5^t$$

$$\therefore\ x_t=k_1+k_2\,2^t+\frac{1}{4}5^t,\quad t=0,1,2,\ldots$$

(f) $x_{t+2}-3x_{t+1}+2x_t=1$

$x_t^{(h)}:\quad x_{t+2}^{(h)}-3x_{t+1}^{(h)}+2x_t^{(h)}=0$

$\therefore\ x_t^{(h)}=k_1+k_2\,2^t$ (ver inciso anterior)

$x_t^{(p)}:\quad x_{t+2}^{(p)}-3x_{t+1}^{(p)}+2x_t^{(p)}=1$

No sirve proponer $x_t^{(p)}=A$ (es l.d. a 1):

$$x_t^{(p)}=A\Longrightarrow A-3A+2A=1\Longrightarrow 0=1$$

Proponemos $x_t^{(p)}=At$

$$\therefore\ x_{t+1}^{p}=A\,(t+1)$$
$$\therefore\ x_{t+2}^{p}=A\,(t+2)$$
$$\therefore\ A\,(t+2)-3A\,(t+1)+2At=1$$
$$\therefore\ A=-1$$
$$\therefore\ x_t^{(p)}=-t$$

$$\therefore\ x_t=k_1+k_2\,2^t-t,\quad t=0,1,2,\ldots$$

(g) $x_{t+2}-3x_{t+1}+2x_t=6(2)^t$

$x_t^{(h)}:\quad x_{t+2}^{(h)}-3x_{t+1}^{(h)}+2x_t^{(h)}=0$

$\therefore\ x_t^{(h)}=k_1+k_2\,2^t$ (ver inciso anterior)

$x_t^{(p)}:\quad x_{t+2}^{(p)}-3x_{t+1}^{(p)}+2x_t^{(p)}=6(2)^t$

No sirve proponer $x_t^{(p)}=A(2)^t$ (es l.d. a 2^t):

$$x_t^{(p)}=A\,(2)^t\Longrightarrow 4\,A\,(2)^t-3\,\left[2A(2)^t\right]+2\,[A(2)^t]=6(2)^t\Longrightarrow 0=6$$

Proponemos $x_t^{(p)} = At(2)^t$

$\therefore\ x_{t+1}^p = A\left(t+1\right)\left(2\right)^{t+1}$

$\therefore\ x_{t+2}^p = A\left(t+2\right)\left(2\right)^{t+2}$

$\therefore\ A\left(t+2\right)\left(2\right)^{t+2} - 3A\left(t+1\right)\left(2\right)^{t+1} + 2At(2)^t = 6(2)^t$

$\therefore\ A = 3$

$\therefore\ x_t^{(p)} = 3t(2)^t$

$\therefore\ x_t = k_1 + k_2\, 2^t + 3t(2)^t, \quad t = 0, 1, 2, \ldots$

(h) $x_{t+2} - 6x_{t+1} + 9x_t = 8 + 3\left(2^t\right)$

$x_t^{(h)}: \quad x_{t+2}^{(h)} - 6x_{t+1}^{(h)} + 9x_t^{(h)} = 0$

Proponemos $x_t^{(h)} = \lambda^t$

$\therefore\ \lambda^2 - 6\lambda + 9 = 0$

$\therefore\ \left(\lambda - 3\right)^2 = 0$

$\therefore\ \lambda_1 = \lambda_2 = 3$

$\therefore\ x_t^{(h)} = k_1\, 3^t + k_2\, t3^t$

$x_t^{(p)}: \quad x_{t+2}^{(p)} - 6x_{t+1}^{(p)} + 9x_t^{(p)} = 8 + 3\left(2^t\right)$

Proponemos $x_t^{(p)} = A + B2^t$

$\therefore\ x_{t+1}^p = A + B2^{t+1}$

$\therefore\ x_{t+2}^p = A + B2^{t+2}$

$\therefore\ \left[A + B2^{t+2}\right] - 6\left[A + B2^{t+1}\right] + 9\left[A + B2^t\right] = 8 + 3\left(2^t\right)$

$\therefore\ 4A + B2^t = 8 + 3\left(2^t\right)$

$\therefore\ A = 2, \ \ B = 3$

$\therefore\ x_t^{(p)} = 2 + 3\left(2^t\right)$

$\therefore\ x_t = k_1\, 3^t + k_2\, t3^t + 2 + 3\left(2^t\right), \quad t = 0, 1, 2, \ldots$

4. (a) $x_{t+1} + 4x_t = 10$, $x_0 = 5$

$x_t^{(h)}: \quad x_{t+1}^{(h)} + 4x_t^{(h)} = 0$

Proponemos $x_t^{(h)} = \lambda^t$

$\therefore\ \lambda + 4 = 0$

$\therefore\ \lambda = -4$

$\therefore\ x_t^{(h)} = A\left(-4\right)^t$ (sol. general, $A \neq x_0$)

$x_t^{(p)}: \quad x_{t+1}^{(p)} + 4x_t^{(p)} = 10$

Proponemos $x_t^{(p)} = B$

$\therefore\ x_{t+1}^{(p)} = B$

$\therefore\ B + 4B = 10$

$\therefore\ B = 2$

$\therefore\ x_{t+1}^{(p)} = 2$

$\therefore\ x_t = A\,(-4)^t + 2$

Condiciones iniciales:

$x_0 = 5 = A + 2$

$\therefore\ A = 3$

$\therefore\ x_t = 3\,(-4)^t + 2, \quad t = 0, 1, 2, \ldots$

Este es el mismo resultado que el obtenido con el método de la tarea 1 ($x_t = a^t(x_0 - x^*) + x^*$).

(b) $x_{t+1} = 2x_t + 9\,(5^t)\,,\ x_0 = 3$

$x_t^{(h)}: \quad x_{t+1}^{(h)} = 2x_t^{(h)}$

$\therefore\ x_t^{(h)} = A\,(2^t)$

$x_t^{(p)}: \quad x_{t+1}^{(p)} = 2x_t^{(p)} + 9\,(5^t)$

Proponemos $x_t^{(p)} = B\,(5^t)$

$\therefore\ x_{t+1}^{(p)} = B\,(5^{t+1})$

$\therefore\ B\,(5^{t+1}) = 2B\,(5^t) + 9\,(5^t)$

$\therefore\ 5B = 2B + 9$

$\therefore\ B = 3$

$\therefore\ x_{t+1}^{(p)} = 3\,(5^t)$

$\therefore\ x_t = A\,(2^t) + 3\,(5^t)$

Condiciones iniciales:

$x_0 = 3 = A + 3$

$\therefore\ A = 0$

$\therefore\ x_t = 3\,(5^t)\,, \quad t = 0, 1, 2, \ldots$

5. $x_{t+3} - 3x_{t+1} + 2x_t = 0$

Proponemos $x_t = \lambda^t$

$\therefore\ \lambda^{t+3} - 3\lambda^{t+1} + 2\lambda^t = 0$

$\therefore\ \lambda^t\left(\lambda^3 - 3\lambda + 2\right) = 0$

$\therefore\ \lambda^3 - 3\lambda + 2 = 0$

$\therefore\ (\lambda - 1)\left(\lambda^2 + \lambda - 2\right) = 0$

$\therefore\ (\lambda - 1)\ (\lambda - 1)\ (\lambda + 2) = 0$

$\therefore\ \lambda_1 = \lambda_2 = 1,\ \ \lambda_3 = -2$ (2 raíces reales repetidas)

$\therefore\ x_t = k_1\,1^t + k_2\,t1^t + k_3\ (-2)^t$

$\therefore\ x_t = k_1 + k_2 t + k_3\ (-2)^t\,, \quad t = 0, 1, 2, \ldots$

6. $2x_{t+2} - 5x_{t+1} + 2x_t = -6$, con $x_0 = 4$ y $x_1 = \beta$.

(a) El punto fijo x^* se obtiene de $2x^* - 5x^* + 2x^* = -6$, de donde $x^* = 6$.

(b) $x_t^{(h)}$: $\quad 2x_{t+2}^{(h)} - 5x_{t+1}^{(h)} + 2x_t^{(h)} = 0$

Proponemos $x_t^{(h)} = \lambda^t$

$$\therefore\ 2\lambda^2 - 5\lambda + 2 = 0$$

$$\therefore\ \lambda_{1,2} = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4}$$

$$\therefore\ \lambda_1 = 2, \quad \lambda_2 = \frac{1}{2}$$

$$\therefore\ x_t^{(h)} = k_1\, 2^t + k_2 \left(\frac{1}{2}\right)^t$$

$x_t^{(p)}$: $\quad 2x_{t+2}^{(p)} - 5x_{t+1}^{(p)} + 2x_t^{(p)} = -6$

Proponemos $x_t^{(p)} = A$

$$\therefore\ 2A - 5A + 2A = -6$$

$$\therefore\ A = 6$$

$$\therefore\ x_t^{(p)} = 6 = x^*$$

$$\therefore\ x_t = k_1\, 2^t + k_2 \left(\frac{1}{2}\right)^t + 6$$

Condiciones iniciales:

$$x_0 = 4 = k_1 + k_2 + 6$$

$$x_1 = \beta = k_1\,(2) + k_2 \left(\frac{1}{2}\right) + 6$$

$$\therefore\ k_1 = \frac{2\beta - 10}{3}, \quad k_2 = \frac{4 - 2\beta}{3}$$

$$\therefore\ x_t = \left(\frac{2\beta - 10}{3}\right)\ 2^t + \left(\frac{4 - 2\beta}{3}\right) \left(\frac{1}{2}\right)^t + 6, \quad t = 0, 1, 2, \ldots$$

(c) Como $\lim\limits_{t\to\infty} 2^t$ diverge, x^* es estable sólo si $k_1 = \dfrac{2\beta - 10}{3} = 0$, esto es, si $\beta = 5$. En ese caso,

$$x_t = -2\left(\frac{1}{2}\right)^t + 6$$

$$\therefore\ \lim_{t\to\infty} x_t = -2 \lim_{t\to\infty} \left(\frac{1}{2}\right)^t + 6 = 6 = x^*$$

(d) Como $\lim_{t\to -\infty}(1/2)^t$ diverge, x^* es inestable sólo si $k_2 = \dfrac{4-2\beta}{3} = 0$, esto es, si $\beta = 2$. En ese caso,

$$x_t = -2\ (2)^t + 6$$

$$\therefore \lim_{t\to -\infty} x_t = -2 \lim_{t\to -\infty} 2^t + 6 = 6 = x^*$$

7. $x_{t+2} - ax_{t+1} + \dfrac{1}{16}x_t = 0$, con a constante.

Proponemos $x_t = \lambda^t$

$$\therefore \lambda^2 - a\lambda + \frac{1}{16} = 0$$

$$\therefore \lambda_{1,2} = \frac{1}{2}\left[a \pm \sqrt{a^2 - \frac{1}{4}}\right]$$

La solución presenta un comportamiento oscilatorio cuando $a^2 < \dfrac{1}{4}$, esto es, cuando $-\dfrac{1}{2} < a < \dfrac{1}{2}$.

En ese caso,

$$\lambda_{1,2} = \frac{1}{2}\left[a \pm \sqrt{(-1)\left(\frac{1}{4} - a^2\right)}\right] = \frac{1}{2}\left[a \pm i\sqrt{\frac{1}{4} - a^2}\right]$$

Reescribiendo $\lambda_{1,2}$ de la forma $\lambda_{1,2} = \alpha \pm i\beta$ se tiene

$$\alpha = \frac{a}{2},\quad \beta = \frac{1}{2}\sqrt{\frac{1}{4} - a^2}$$

$$\|\lambda\| = \sqrt{\alpha^2 + \beta^2} = \frac{1}{4},\quad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$\therefore x_t = \left(\frac{1}{4}\right)^t \left[k_1 \cos(\theta t) + k_2\, sen\,(\theta t)\right],\quad t = 0, 1, 2, \ldots$$

8. $F_{t+2} = F_{t+1} + F_t, \quad F_0 = 0,\ F_1 = 1$

Solución:

Proponemos $F_t = \lambda^t$

$$\therefore \lambda^2 - \lambda - 1 = 0$$

$$\therefore \lambda_1 = \frac{1+\sqrt{5}}{2},\ \lambda_2 = \frac{1-\sqrt{5}}{2} \quad \text{(raíces reales distintas)}$$

$$\therefore F_t = k_1\left(\frac{1+\sqrt{5}}{2}\right)^t + k_2\left(\frac{1-\sqrt{5}}{2}\right)^t$$

Condiciones iniciales:

$F_0 = 0 = k_1 + k_2$

$F_1 = 1 = \left(\dfrac{1+\sqrt{5}}{2}\right) k_1 + \left(\dfrac{1-\sqrt{5}}{2}\right) k_2$

$\therefore\ k_1 = \dfrac{1}{\sqrt{5}},\ \ k_2 = -\dfrac{1}{\sqrt{5}}$

$\therefore\ F_t = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^t - \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^t, \quad t = 0, 1, 2, \ldots$

9. $C_t = cY_{t-1}, \quad K_t = \sigma Y_{t-1}, \quad Y_t = C_t + K_t - K_{t-1},\ \ c, \sigma > 0$

$\therefore\ Y_t = (cY_{t-1}) + (\sigma Y_{t-1}) - (\sigma Y_{t-2})$

$\therefore\ Y_t = (c+\sigma)\, Y_{t-1} - \sigma Y_{t-2}$

$\therefore\ Y_t - (c+\sigma)\, Y_{t-1} + \sigma Y_{t-2} = 0$

o bien, $Y_{t+2} - (c+\sigma)\, Y_{t+1} + \sigma Y_t = 0$

Ecuación:

$Y_{t+2} - (c+\sigma)\, Y_{t+1} + \sigma Y_t = 0$

Solución:

Proponemos $Y_t = \lambda^t$

$\therefore\ \lambda^2 - (c+\sigma)\,\lambda + \sigma = 0$

$\therefore\ \lambda_{1,2} = \dfrac{(c+\sigma) \pm \sqrt{(c+\sigma)^2 - 4\sigma}}{2}$

i) Si $(c+\sigma)^2 > 4\sigma$, entonces λ_1 y λ_2 son reales y distintas

$\therefore\ Y_t = k_1\lambda_1^t + k_2\lambda_2^t, \quad t = 0, 1, 2, \ldots$

ii) Si $(c+\sigma)^2 = 4\sigma$, entonces $\lambda_1 = \lambda_2 = \dfrac{c+\sigma}{2}$ son reales repetidas

$\therefore\ Y_t = k_1\left(\dfrac{c+\sigma}{2}\right)^t + k_2\, t\left(\dfrac{c+\sigma}{2}\right)^t, \quad t = 0, 1, 2, \ldots$

iii) Si $(c+\sigma)^2 < 4\sigma$, entonces λ_1 y λ_2 son complejas

En este caso, $\lambda_{1,2} = \dfrac{(c+\sigma) \pm \sqrt{-\left[4\sigma - (c+\sigma)^2\right]}}{2} = \dfrac{(c+\sigma) \pm i\sqrt{4\sigma - (c+\sigma)^2}}{2}$

Reescribiendo $\lambda_{1,2}$ de la forma $\lambda_{1,2} = \alpha \pm i\beta$ se tiene

$\alpha = \dfrac{c+\sigma}{2},\ \ \beta = \dfrac{\sqrt{4\sigma - (c+\sigma)^2}}{2}$

$$\|\lambda\| = \sqrt{\alpha^2+\beta^2} = \sqrt{\sigma}, \quad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$\therefore\ Y_t = \sigma^{t/2}\left[k_1\cos(\theta t) + k_2\, sen\,(\theta t)\right], \quad t = 0,1,2,\ldots$$

10. $Y_t = C_t + I_t + G_t,\ \ C_t = C_0 + \alpha Y_{t-1},\ \ I_t = I_0 + \beta\left(Y_{t-1} - Y_{t-2}\right),$
con $0 < \alpha < 1$, $C_0, I_0, \beta > 0$, $(\alpha+\beta)^2 > 4\beta$

$$\therefore\ Y_t = (C_0 + \alpha Y_{t-1}) + (I_0 + \beta(Y_{t-1} - Y_{t-2})) + G_t$$

$$\therefore\ Y_t = (\alpha+\beta)\,Y_{t-1} - \beta Y_{t-2} + (C_0 + I_0 + G_t)$$

Suponiendo que $G_t = G$, se tiene

$$Y_t - (\alpha+\beta)\,Y_{t-1} + \beta Y_{t-2} = C_0 + I_0 + G$$

o bien, $Y_{t+2} - (\alpha+\beta)\,Y_{t+1} + \beta Y_t = C_0 + I_0 + G$

Ecuación:

$$Y_{t+2} - (\alpha+\beta)\,Y_{t+1} + \beta Y_t = C_0 + I_0 + G$$

Solución:

$Y_t^{(h)}:\quad Y_{t+2}^{(h)} - (\alpha+\beta)\,Y_{t+1}^{(h)} + \beta Y_t^{(h)} = 0$

Proponemos $Y_t^{(h)} = \lambda^t$

$$\therefore\ \lambda^2 - (\alpha+\beta)\,\lambda + \beta = 0$$

$$\therefore\ \lambda_{1,2} = \frac{(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4\beta}}{2}$$

Como $(\alpha+\beta)^2 > 4\beta$, entonces λ_1 y λ_2 son reales y distintas

$$\therefore\ Y_t^{(h)} = k_1\lambda_1^t + k_2\lambda_2^t$$

$Y_t^{(p)}:\quad Y_{t+2}^{(p)} - (\alpha+\beta)\,Y_{t+1}^{(p)} + \beta Y_t^{(p)} = C_0 + I_0 + G$

Proponemos $Y_t^{(p)} = A$

$$\therefore\ Y_{t+1}^{(p)} = Y_{t+2}^{(p)} = A$$

$$\therefore\ A - (\alpha+\beta)\,A + \beta A = C_0 + I_0 + G$$

$$\therefore\ A = \frac{C_0 + I_0 + G}{1-\alpha}$$

$$\therefore\ Y_t^{(p)} = \frac{C_0 + I_0 + G}{1-\alpha}$$

$$\therefore\ Y_t = k_1\lambda_1^t + k_2\lambda_2^t + \frac{C_0 + I_0 + G}{1-\alpha}, \quad t = 0,1,2,\ldots$$

MATEMÁTICAS APLICADAS A LA ECONOMÍA
TAREA 3 - SOLUCIONES
ELEMENTOS DE PROGRAMACIÓN DINÁMICA
(Temas 3.1-3.3)

1. (a) Sea $\beta \neq 1$. Partimos de la suma geométrica

$$\sum_{k=0}^{n} \beta^k = \frac{1-\beta^{n+1}}{1-\beta}.$$

Para obtener la suma $\sum_{k=0}^{n} k\beta^k$ derivamos con respecto a β ambos lados de la geométrica, esto es,

$$\begin{aligned}
\frac{d}{d\beta}\sum_{k=0}^{n} \beta^k &= \frac{d}{d\beta}\frac{1-\beta^{n+1}}{1-\beta} \\
\sum_{k=0}^{n} \frac{d\beta^k}{d\beta} &= \frac{(1-\beta)\left(-(n+1)\beta^n\right) - \left(1-\beta^{n+1}\right)(-1)}{(1-\beta)^2} \\
\sum_{k=0}^{n} k\beta^{k-1} &= \frac{-(n+1)(1-\beta)\beta^n + \left(1-\beta^{n+1}\right)}{(1-\beta)^2} \\
\frac{1}{\beta}\sum_{k=0}^{n} k\beta^k &= \frac{1-(n+1)\beta^n + n\beta^{n+1}}{(1-\beta)^2} \\
\sum_{k=0}^{n} k\beta^k &= \frac{\beta\left[1-(n+1)\beta^n + n\beta^{n+1}\right]}{(1-\beta)^2}.
\end{aligned}$$

(b) Para obtener la serie aritmético-geométrica, tomamos el límite $n \to \infty$ en el resultado del inciso anterior. Como $|\beta| < 1$, se tiene

$$\lim_{n\to\infty} \beta^n = 0.$$

Por otra parte, por regla de L'Hopital, se tiene

$$\lim_{n\to\infty} n\beta^n = \lim_{n\to\infty} \frac{n}{\beta^{-n}} \overset{L}{=} \lim_{n\to\infty} \frac{1}{-\beta^{-n}\ln\beta} = -\frac{1}{\ln\beta}\lim_{n\to\infty}\beta^n = 0.$$

De esta manera,

$$\begin{aligned}
\sum_{k=0}^{\infty} k\beta^k = \lim_{n\to\infty}\sum_{k=0}^{n} k\beta^k &= \lim_{n\to\infty} \frac{\beta\left[1-(n+1)\beta^n + n\beta^{n+1}\right]}{(1-\beta)^2} \\
&= \frac{\beta}{(1-\beta)^2}\left[1 - \lim_{n\to\infty}(n+1)\beta^n + \lim_{n\to\infty} n\beta^{n+1}\right] \\
&= \frac{\beta}{(1-\beta)^2}.
\end{aligned}$$

2. $V_0(w_0) = \max_{\{c_t\}} \sum_{k=0}^{T} \beta^k \ln c_k$ s.a. $w_{t+1} = w_t - c_t$, $w_0 = \phi$ y $w_T = 0$.

Función valor a partir del período t :

$$V_t(w_t) = \max_{\{c_t\}} \sum_{k=t}^{T} \beta^k \ln c_k, \quad t = 0, 1, 2, \ldots, T.$$

Ecuación de Bellman:

$$V_t(w_t) = \max_{\{c_t\}} \left[\beta^t \ln c_t + V_{t+1}(w_{t+1})\right].$$

Condiciones de primer orden:

La variable de estado es w_t y la de control es c_t. Sean

$$f_t(w_t, c_t) = \beta^t \ln c_t, \quad g_t(w_t, c_t) = w_t - c_t.$$

De esta manera, las condiciones de primer orden

$$\frac{\partial f_t}{\partial c_t} + V'_{t+1}(w_{t+1}) \frac{\partial g_t}{\partial c_t} = 0,$$

$$\frac{\partial f_t}{\partial w_t} + V'_{t+1}(w_{t+1}) \frac{\partial g_t}{\partial w_t} = V'_t(w_t),$$

$$w_{t+1} = w_t - c_t,$$

se reducen a

$$\frac{\beta^t}{c_t} - V'_{t+1}(w_{t+1}) = 0, \text{..........} (1)$$

$$V'_{t+1}(w_{t+1}) = V'_t(w_t), \text{..........} (2)$$

$$w_{t+1} = w_t - c_t \text{..........} (3)$$

Ecuación de Euler y su solución:

De la ec. (1):

$$V'_{t+1}(w_{t+1}) = \frac{\beta^t}{c_t} \text{..........} (4)$$

Sustituimos (4) en (2):

$$\frac{\beta^t}{c_t} = V'_t(w_t) \text{..........} (5)$$

Iteramos (5) un período hacia adelante:

$$\frac{\beta^{t+1}}{c_{t+1}} = V'_{t+1}(w_{t+1}) \text{..........} (6)$$

Sustituimos (6) en (1):

$$\frac{\beta^t}{c_t} - \frac{\beta^{t+1}}{c_{t+1}} = 0$$

de donde se obtiene la ecuación de Euler:

$c_{t+1} = \beta c_t$..(7)

Esta es una ecuación lineal y homogénea. Su solución es

$c_t = c_0 \beta^t$..(8)

Ecuación de restricción para w_t y su solución:

De (3) y (8) se obtiene la ecuación lineal no homogénea:

$w_{t+1} = w_t - c_0 \beta^t$..(9)

La ecuación no es autónoma. Podemos resolverla como sigue:

a) Método 1 (por iteración):

Iterando la solución a partir de $w_0 = \phi$, se obtiene

$$w_t = \phi - c_0 \sum_{k=0}^{t-1} \beta^k = \phi - c_0 \frac{1-\beta^t}{1-\beta}$$

$$= \left[\phi - \frac{c_0}{1-\beta} \right] + \frac{c_0}{1-\beta}\ \beta^t \text{.......................................(10)}$$

Condición de transversalidad:

No hay condición de transversalidad, ya que $w_T = 0$:

$$0 = \left[\phi - \frac{c_0}{1-\beta} \right] + \frac{c_0}{1-\beta} \beta^T \ \therefore\ c_0 = \phi \left(\frac{1-\beta}{1-\beta^T} \right) \text{......(11)}$$

$$\therefore\ w_t = \left[\phi - \frac{\phi}{1-\beta^T} \right] + \frac{\phi}{1-\beta^T}\ \beta^t = \phi \left(\frac{\beta^t - \beta^T}{1-\beta^T} \right) \text{......(12)}$$

b) Método 2 (coeficientes indeterminados):

$$w_t = w_t^{(h)} + w_t^{(p)}$$

$$w_t^{(h)} : \quad w_{t+1}^{(h)} = w_t^{(h)}$$

$$\therefore\ w_t^{(h)} = A$$

$$w_t^{(p)} : \quad w_{t+1}^{(p)} = w_t^{(p)} - c_0 \beta^t$$

Proponemos $w_t^{(p)} = K\beta^t$

$$\therefore\ K\beta^{t+1} = K\beta^t - c_0 \beta^t$$

$$\therefore\ K\beta = K - c_0 \qquad \therefore\ K = \frac{c_0}{1-\beta}$$

$$\therefore\ w_{t+1}^{(p)} = \frac{c_0}{1-\beta}\ \beta^t$$

$$\therefore\ w_t = A + \frac{c_0}{1-\beta}\ \beta^t$$

Condición inicial ($w_0 = \phi$):

$$\phi = A + \frac{c_0}{1-\beta} \qquad \therefore\ A = \phi - \frac{c_0}{1-\beta}$$

$$\therefore\ w_t = \left[\phi - \frac{c_0}{1-\beta}\right] + \frac{c_0}{1-\beta}\ \beta^t$$

que coincide con (10).

Trayectorias óptimas:

De (8), (11) y (12) se se concluye:

$$c_t = \phi\left(\frac{1-\beta}{1-\beta^T}\right)\beta^t \text{ ....}(13)$$

$$w_t = \phi\left(\frac{\beta^t-\beta^T}{1-\beta^T}\right) \text{ ....}(14)$$

3. $V_0(w_0) = \max\limits_{\{c_t\}} \sum\limits_{k=0}^{\infty} \beta^k\ u(c_k)$ s.a. $w_{t+1} = (1+r)(w_t - c_t)$, w_0 dado.

Función valor a partir del período t :

$$V_t(w_t) = \max_{\{c_t\}} \sum_{k=t}^{\infty} \beta^k\ u(c_k)\,,\ \ t = 0, 1, 2, \ldots$$

Ecuación de Bellman:

$$V_t(w_t) = \max_{\{c_t\}} \left[\beta^t\ u(c_t) + V_{t+1}(w_{t+1})\right].$$

Condiciones de primer orden:

La variable de estado es w_t y la de control es c_t. Sean

$$f_t(w_t, c_t) = \beta^t u(c_t)\,,\ \ g_t(w_t, c_t) = (1+r)(w_t - c_t)\,.$$

De esta manera, las condiciones de primer orden

$$\frac{\partial f_t}{\partial c_t} + V'_{t+1}(w_{t+1})\frac{\partial g_t}{\partial c_t} = 0,$$

$$\frac{\partial f_t}{\partial w_t} + V'_{t+1}(w_{t+1})\frac{\partial g_t}{\partial w_t} = V'_t(w_t)\,,$$

$$w_{t+1} = (1+r)(w_t - c_t)\,,$$

se reducen a

$$\beta^t u'(c_t) - (1+r)V'_{t+1}(w_{t+1}) = 0, \text{.......................................}(1)$$

$$(1+r)V'_{t+1}(w_{t+1}) = V'_t(w_t)\,, \text{...}(2)$$

$$w_{t+1} = (1+r)(w_t - c_t) \text{ ....}(3)$$

Ecuación de Euler:

De la ec. (1):

$$V'_{t+1}(w_{t+1}) = \frac{\beta^t u'(c_t)}{1+r} \text{.......................................}(4)$$

Sustituimos (4) en (2):

$$\beta^t u'(c_t) = V'_t(w_t) \text{.......................................}(5)$$

Iteramos (5) un período hacia adelante:

$$\beta^{t+1} u'(c_{t+1}) = V'_{t+1}(w_{t+1}) \text{.......................................}(6)$$

Sustituimos (6) en (1):

$$\beta^t u'(c_t) - (1+r)\beta^{t+1} u'(c_{t+1}) = 0$$

de donde se obtiene la ecuación de Euler:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1+r) \text{.......................................}(7)$$

4. De acuerdo con el problema 3, se tiene $u'(c_t)/u'(c_{t+1}) = \beta(1+r)$.

Sea

$$u(c_t) = c_t^\alpha \text{.......................................}(1)$$

Por lo tanto,

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{\alpha c_t^{\alpha-1}}{\alpha c_{t+1}^{\alpha-1}} = \beta(1+r).$$

Así, la ecuación de Euler para este modelo es

$$\frac{c_{t+1}}{c_t} = [\beta(1+r)]^{\frac{1}{1-\alpha}} \text{.......................................}(2)$$

Por simplicidad, definimos

$$\gamma = [\beta(1+r)]^{\frac{1}{1-\alpha}}, \text{.......................................}(3)$$

con $0 < \gamma < 1$. Así, la ecuación (2) se convierte en

$$c_{t+1} = \gamma c_t, \quad \text{(ec. lineal homogénea)}$$

$$\therefore \quad c_t = c_0 \gamma^t \text{.......................................}(4)$$

Ecuación de restricción para w_t:

$$w_{t+1} = (1+r)(w_t - c_t)$$

$$\therefore w_{t+1} = (1+r) w_t - c_0 (1+r) \gamma^t \quad \text{(ec. lineal no homogénea)}$$

La ecuación no es autónoma. Podemos resolverla como sigue:

a) Método 1 (coeficientes indeterminados):

$$w_t = w_t^{(h)} + w_t^{(p)}$$

$w_t^{(h)}: \quad w_{t+1}^{(h)} - (1+r)\,w_t^{(h)} = 0$

Proponemos $w_t^{(h)} = \lambda^t$

$\therefore\ \lambda - (1+r) = 0$

$\therefore\ \lambda = 1+r$

$\therefore\ w_t^{(h)} = A\,(1+r)^t$ (sol. general, $A \neq w_0$)

$w_t^{(p)}: \quad w_{t+1}^{(p)} = (1+r)\,w_t^{(p)} - c_0\,(1+r)\,\gamma^t$

Proponemos $w_t^{(p)} = B\gamma^t$

$\therefore\ B\gamma^{t+1} = (1+r)\,B\gamma^t - c_0\,(1+r)\,\gamma^t$

$\therefore\ B\gamma = (1+r)\,B - c_0\,(1+r)$

$$\therefore\ B = \frac{c_0(1+r)}{1+r-\gamma}$$

$$\therefore\ w_{t+1}^{(p)} = \frac{c_0(1+r)}{1+r-\gamma}\ \gamma^t$$

$$\therefore\ w_t = A\,(1+r)^t + \frac{c_0(1+r)}{1+r-\gamma}\ \gamma^t$$

Condiciones iniciales:

$$w_0 = A + \frac{c_0(1+r)}{1+r-\gamma} \qquad \therefore\ A = w_0 - \frac{c_0(1+r)}{1+r-\gamma}$$

$$\therefore\ w_t = \left[w_0 - \frac{c_0(1+r)}{1+r-\gamma}\right](1+r)^t + \frac{c_0(1+r)}{1+r-\gamma}\ \gamma^t \text{..............} (5)$$

b) Método 2 (por iteración):

Iterando la solución se obtiene

$$w_t = (1+r)^t\,w_0 + \sum_{k=0}^{t-1}(1+r)^{t-k-1}\left[-c_0\,(1+r)\,\gamma^k\right]$$

$$= (1+r)^t\,w_0 - c_0\,(1+r)^t \sum_{k=0}^{t-1}\left(\frac{\gamma}{1+r}\right)^k$$

$$= (1+r)^t\,w_0 - c_0\,(1+r)^t\ \frac{1-\left(\dfrac{\gamma}{1+r}\right)^t}{1-\left(\dfrac{\gamma}{1+r}\right)}$$

$$\therefore\ w_t = \left[w_0 - \frac{c_0(1+r)}{1+r-\gamma}\right](1+r)^t + \frac{c_0(1+r)}{1+r-\gamma}\ \gamma^t,$$

que coincide con la solución (5).

Funciones w_t y c_t :

Como $1+r>1$, la solución (5) converge a la larga sólo si

$$w_0 - \frac{c_0(1+r)}{1+r-\gamma} = 0$$

$$\therefore\ c_0 = \frac{w_0\,(1+r-\gamma)}{1+r} = w_0\left(1-\frac{\gamma}{1+r}\right) \text{..........................(6)}$$

Sustituyendo (6) en (4) y (5), concluimos que

$$c_t = w_0\left(1-\frac{\gamma}{1+r}\right)\gamma^t \text{...(7)}$$

$$w_t = w_0\gamma^t \text{..(8)}$$

Función valor para $\alpha = 1/2$:

Cuando $\alpha = 1/2$, γ en (3) se convierte en

$$\gamma = [\beta\,(1+r)]^2 \text{...(9)}$$

En ese caso, las soluciones (7) y (8) están dadas por

$$c_t = w_0\left[1-\beta^2\,(1+r)\right][\beta\,(1+r)]^{2t} \text{.............................(10)}$$

$$w_t = w_0\,[\beta\,(1+r)]^{2t}\ , \text{..(11)}$$

Sustituyendo (10) en (1), se tiene

$$u\,(c_t) = \sqrt{w_0}\sqrt{1-\beta^2\,(1+r)}\,[\beta\,(1+r)]^t \text{......................(12)}$$

Como $V_0(w_0) = \sum_{k=0}^{\infty}\ \beta^k u\,(c_k)$, por lo tanto,

$$V_0(w_0) = \sum_{k=0}^{\infty}\beta^k\ \sqrt{w_0}\sqrt{1-\beta^2\,(1+r)}\ \,[\beta\,(1+r)]^k$$

$$= \sqrt{w_0}\sqrt{1-\beta^2\,(1+r)}\sum_{k=0}^{\infty}\ \left[\beta^2\,(1+r)\right]^k .$$

Usando el resultado del ejercicio 8b de la tarea 1 se obtiene

$$V_0(w_0) = \sqrt{w_0}\sqrt{1-\beta^2\,(1+r)}\left(\frac{1}{1-\beta^2\,(1+r)}\right)$$

$$= \sqrt{\frac{w_0}{1-\beta^2\,(1+r)}} \text{...(13)}$$

5. De acuerdo con el problema 3, se tiene $u'\,(c_t)\,/u'\,(c_{t+1}) = \beta\,(1+r)$.

   Ecuación de Euler para c_t:

   Sea

$$u\,(c_t) = \ln c_t \text{..(1)}$$

Por lo tanto,

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} = \beta(1+r).$$

Así, la ecuación de Euler para este modelo es

$$c_{t+1} = \beta(1+r)\,c_t \text{...}(2)$$

$$\therefore c_t = c_0\left[\beta(1+r)\right]^t \text{...}(3)$$

Ecuación de restricción para w_t:

$$w_{t+1} = (1+r)(w_t - c_t)$$

$$\therefore w_{t+1} = (1+r)\,w_t - c_0\beta^t(1+r)^{t+1} \quad \text{(ec. lineal no homogénea)}$$

Procediendo simillarmente al ejercicio 4, se obtiene

$$w_t = \left[w_0 - \frac{c_0}{1-\beta}\right](1+r)^t + \frac{c_0}{1-\beta}\left[\beta(1+r)\right]^t \text{................}(4)$$

Funciones w_t y c_t :

La solución (4) converge a la larga sólo si $w_0 - \dfrac{c_0}{1-\beta} = 0$,

de donde

$$c_0 = w_0(1-\beta) \text{...}(5)$$

Sustituyendo (5) en (3) y (4), concluimos que

$$c_t = w_0(1-\beta)\left[\beta(1+r)\right]^t \text{..}(6)$$

$$w_t = w_0\left[\beta(1+r)\right]^t \text{...}(7)$$

Función valor:

Sustituyendo (6) en (1), se tiene

$$u(c_t) = \ln\left[w_0(1-\beta)\left[\beta(1+r)\right]^t\right]$$

$$= \ln\left[w_0(1-\beta)\right] + t\ln\left[\beta(1+r)\right] \text{..........................}(8)$$

Como $V_0(w_0) = \sum_{k=0}^{\infty} \beta^k u(c_t)$, por lo tanto

$$V_0(w_0) = \sum_{k=0}^{\infty} \beta^k \left\{\ln\left[w_0(1-\beta)\right] + k\ln\left[\beta(1+r)\right]\right\}$$

$$V_0(w_0) = \ln\left[w_0(1-\beta)\right] \sum_{k=0}^{\infty} \beta^k + \ln\left[\beta(1+r)\right] \sum_{k=0}^{\infty} k\beta^k$$

De acuerdo con el resultado del ejercicio 1b se obtiene

$$V_0(w_0) = \ln\left[w_0(1-\beta)\right] \frac{1}{1-\beta} + \ln\left[\beta(1+r)\right] \frac{\beta}{(1-\beta)^2} \text{.....}(9)$$

6. De acuerdo con el problema 3, se tiene $u'(c_t)/u'(c_{t+1}) = \beta(1+r)$.

   Ecuación de Euler para c_t:

   Sea

   $$u(c_t) = 1 - \frac{e^{-ac_t}}{a} \quad \text{..........(1)}$$

   Por lo tanto,

   $$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{e^{-ac_t}}{e^{-ac_{t+1}}} = e^{a(c_{t+1}-c_t)} = \beta(1+r).$$

   Así, la ecuación de Euler para este modelo es

   $$c_{t+1} = c_t + \tfrac{1}{a}\ln[\beta(1+r)] \quad \text{..........(2)}$$

   Se trata de una ecuación lineal autónoma, no homogénea.

   Se resuelve más fácilmente por iteración, obteniendo

   $$c_t = c_0 + \tfrac{t}{a}\ln[\beta(1+r)] \quad \text{..........(3)}$$

7. $V_0(x_0) = \max_{\{c_t\}} \sum_{k=0}^{\infty} \beta^k \ln(x_k - c_k), \quad 0 < \beta < 1$

   s.a. $x_{t+1} = c_t$, x_0 dado, $\lim_{t\to\infty} c_t = 0$.

   Función valor a partir del período t :

   $$V_t(x_t) = \max_{\{c_t\}} \sum_{k=t}^{\infty} \beta^k \ln(x_k - c_k), \quad t = 0, 1, 2, \ldots$$

   Ecuación de Bellman:

   $$V_t(x_t) = \max_{\{c_t\}} \left[\beta^t \ln(x_t - c_t) + V_{t+1}(x_{t+1})\right].$$

   Condiciones de primer orden:

   Sean

   $$f_t(k_t, c_t) = \beta^t \ln(x_t - c_t), \quad g_t(x_t, c_t) = c_t.$$

   De esta manera, las condiciones de primer orden

   $$\frac{\partial f_t}{\partial c_t} + V'_{t+1}(x_{t+1})\frac{\partial g_t}{\partial c_t} = 0$$

   $$\frac{\partial f_t}{\partial x_t} + V'_{t+1}(x_{t+1})\frac{\partial g_t}{\partial x_t} = V'_t(x_t)$$

   $$x_{t+1} = c_t$$

   se reducen a

   $$-\frac{\beta^t}{x_t - c_t} + V'_{t+1}(x_{t+1}) = 0 \quad \text{..........(1)}$$

$$\frac{\beta^t}{x_t - c_t} = V_t'(x_t) \text{..}(2)$$

$$x_{t+1} = c_t \text{..}(3)$$

Ecuación de Euler:

Iteramos (2) un período hacia adelante:

$$V_{t+1}'(x_{t+1}) = \frac{\beta^{t+1}}{x_{t+1} - c_{t+1}} \text{..}(4)$$

Sustituimos (4) en (1):

$$-\frac{\beta^t}{x_t - c_t} + \frac{\beta^{t+1}}{x_{t+1} - c_{t+1}} = 0, \text{..}(5)$$

de donde se obtiene la ecuación de Euler:

$$x_{t+1} - c_{t+1} = \beta (x_t - c_t) \text{..}(6)$$

Solución:

a) Método 1 (con ecuación de segundo orden):

Sustituyendo (3) en (6) se obtiene

$$x_{t+1} - x_{t+2} = \beta (x_t - x_{t+1})$$

esto es,

$$x_{t+2} - (\beta + 1) x_{t+1} + \beta x_t = 0.$$

Proponemos $x_t = \lambda^t$

$$\therefore \lambda^2 - (\beta + 1) \lambda + \beta = 0$$

$$\therefore \lambda_{1,2} = \frac{(\beta+1) \pm \sqrt{(\beta+1)^2 - 4\beta}}{2} = \frac{(\beta+1) \pm \sqrt{(\beta-1)^2}}{2}$$

$$\therefore \lambda_1 = 1, \quad \lambda_2 = \beta$$

$$\therefore x_t = k_1 + k_2 \beta^t \text{..}(7)$$

$$\therefore c_t = x_{t+1} = k_1 + k_2 \beta^{t+1} \text{..}(8)$$

Por último, como $0 < \beta < 1$ se tiene:

$$\lim_{t \to \infty} c_t = 0 \implies k_1 = 0$$

$$x_0 \text{ dado} \implies k_2 = x_0$$

De este modo, la solución es

$$x_t = x_0 \beta^t, \text{..}(9)$$

$$c_t = x_0 \beta^{t+1}. \text{..}(10)$$

b) Método 2 (por iteración):

Resolviendo (6) por iteración, se obtiene

$$x_t - c_t = \beta^t (x_0 - c_0) \,. \quad \text{..}(11)$$

Sustituyendo (3) en (11), se llega a que

$$x_t - x_{t+1} = \beta^t (x_0 - c_0) \,, \quad \text{..}(12)$$

cuya solución (por iteración o con $x_t^{(h)} + x_t^{(p)}$) es

$$x_t = \left(x_0 - \frac{x_0 - c_0}{1-\beta} \right) + \left(\frac{x_0 - c_0}{1-\beta} \right) \beta^t, \quad \text{..............................}(13)$$

$$c_t = \left(x_0 - \frac{x_0 - c_0}{1-\beta} \right) + \left(\frac{x_0 - c_0}{1-\beta} \right) \beta^{t+1}. \quad \text{........................}(14)$$

Como $0 < \beta < 1$, se tiene:

$$\lim_{t \to \infty} c_t = 0 \Longrightarrow c_0 = \beta x_0.$$

De este modo, la solución es

$$x_t = x_0 \beta^t$$

$$c_t = x_0 \beta^{t+1},$$

que coincide con (9) y (10).

Por último, como $0 < \beta < 1$ y $x_0 > 0$, claramente se satisface

$$x_t - c_t = x_0 \beta^t (1 - \beta) > 0.$$

8. $V_0(k_0) = \max\limits_{\{c_t\}} \sum\limits_{i=0}^{\infty} \beta^i \ln c_i$

s.a. $k_{t+1} = k_t^\alpha - c_t, \quad k_0$ dado.

Función valor a partir del período t :

$$V_t(k_t) = \max_{\{c_t\}} \sum_{i=t}^{\infty} \beta^i \ln c_i, \quad t = 0, 1, 2, \ldots$$

Ecuación de Bellman:

$$V_t(k_t) = \max_{\{c_t\}} \left[\beta^t \ln c_t + V_{t+1}(k_{t+1}) \right].$$

Condiciones de primer orden:

Sean

$$f_t(k_t, c_t) = \beta^t \ln c_t, \qquad g_t(k_t, c_t) = k_t^\alpha - c_t.$$

De esta manera, las condiciones de primer orden

$$\frac{\partial f_t}{\partial c_t} + V'_{t+1}(k_{t+1}) \frac{\partial g_t}{\partial c_t} = 0$$
$$\frac{\partial f_t}{\partial k_t} + V'_{t+1}(k_{t+1}) \frac{\partial g_t}{\partial k_t} = V'_t(k_t)$$
$$k_{t+1} = k_t^{\alpha} - c_t$$

se reducen a

$$\frac{\beta^t}{c_t} - V'_{t+1}(k_{t+1}) = 0 \text{.........} (1)$$
$$V'_{t+1}(k_{t+1}) \left(\alpha k_t^{\alpha-1}\right) = V'_t(k_t) \text{.........} (2)$$
$$k_{t+1} = k_t^{\alpha} - c_t \text{.........} (3)$$

Ecuación de Euler:

De la ec. (1):

$$V'_{t+1}(k_{t+1}) = \frac{\beta^t}{c_t} \text{.........} (4)$$

Sustituimos (4) en (2):

$$\frac{\beta^t}{c_t} \left(\alpha k_t^{\alpha-1}\right) = V'_t(k_t) \text{.........} (5)$$

Iteramos (5) un período hacia adelante:

$$\frac{\beta^{t+1}}{c_{t+1}} \left(\alpha k_{t+1}^{\alpha-1}\right) = V'_{t+1}(k_{t+1}) \text{.........} (6)$$

Sustituimos (6) en (1):

$$\frac{\beta^t}{c_t} - \frac{\alpha\beta^{t+1}}{c_{t+1}} k_{t+1}^{\alpha-1} = 0$$

de donde se obtiene la ecuación de Euler:

$$\frac{c_{t+1}}{c_t} = \alpha\beta k_{t+1}^{\alpha-1} \text{.........} (7)$$

Puntos fijos ($k_t = k^*$ y $c_t = c^*$):

Sustituyendo $k_t = k^*$ y $c_t = c^*$ en (3) y (7) se obtiene

$$k^* = (k^*)^{\alpha} - c^*$$
$$1 = \alpha\beta (k^*)^{\alpha-1},$$

de donde se obtienen los puntos fijos del sistema:

$$k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$$
$$c^* = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}.$$

MATEMÁTICAS APLICADAS A LA ECONOMÍA
TAREA 4 - SOLUCIONES
ECUACIONES DIFERENCIALES I
(PRIMERA PARTE)
(Temas 4.1-4.2)

1. (a) $\dot{x} = 3x + 2e^{-t},\ x(0) = 1; \quad x(t) = Ce^{3t} - \frac{1}{2}e^{-t}$

$x = Ce^{3t} - \frac{1}{2}e^{-t} \implies \dot{x} = 3Ce^{3t} + \frac{1}{2}e^{-t}$

$\therefore\ \dot{x} - 3x = \left(3Ce^{3t} + \frac{1}{2}e^{-t}\right) - 3\left(Ce^{3t} - \frac{1}{2}e^{-t}\right)$

$\therefore\ \dot{x} - 3x = 2e^{-t}$

$\therefore\ \dot{x} = 3x + 2e^{-t}$

Condición inicial:

$x(t) = Ce^{3t} - \frac{1}{2}e^{-t}$, con $x(0) = 1$

$\therefore\ 1 = C - \frac{1}{2} \quad \therefore\ C = \frac{3}{2}$

Solución:

$x(t) = \frac{3}{2}e^{3t} - \frac{1}{2}e^{-t}$

(b) $\dot{x} + 2tx^2 = 0,\ x(1) = 3; \quad x(t) = \frac{1}{t^2 + C}$

$x = \frac{1}{t^2 + C} \implies \dot{x} = -\frac{2t}{(t^2 + C)^2}$

$\therefore\ \dot{x} + 2tx^2 = -\frac{2t}{(t^2 + C)^2} + 2t\left(\frac{1}{t^2 + C}\right)^2$

$\therefore\ \dot{x} + 2tx^2 = 0$

Condición inicial:

$x(t) = \frac{1}{t^2 + C}$, con $x(1) = 3$

$\therefore\ 3 = \frac{1}{1 + C} \quad \therefore\ C = -\frac{2}{3}$

Solución:

$x(t) = \frac{1}{t^2 - (2/3)} = \frac{3}{3t^2 - 2}$

(c) $\dot{x} = 3t^2\left(x^2 + 1\right),\ x(0) = 1; \quad x(t) = \tan\left(t^3 + C\right)$

$x = \tan\left(t^3 + C\right) \implies \dot{x} = 3t^2 \sec^2\left(t^3 + C\right)$

$\therefore\ 3t^2\left(x^2 + 1\right) = 3t^2\left[\tan^2\left(t^3 + C\right) + 1\right]$

Como $\tan^2\theta + 1 = \sec^2\theta$, por lo tanto

$3t^2\left(x^2 + 1\right) = 3t^2 \sec^2\left(t^3 + C\right)$

$\therefore\ 3t^2\left(x^2+1\right)=\dot{x}$

Condición inicial:

$x(t)=\tan\left(t^3+C\right)$, con $x(0)=1$

$\therefore\ 1=\tan(0+C)=\tan C \quad \therefore\ C=\tan^{-1}(1)=\dfrac{\pi}{4}$

Solución:

$x(t)=\tan\left(t^3+\dfrac{\pi}{4}\right)$

2. (a) $\dfrac{dv}{dt}\propto 250-v$

   Para eliminar el símbolo de proporcionalidad $\propto$ introducimos una constante k :

   $\dfrac{dv}{dt}=k\,(250-v).$

   (b) $\dfrac{dN}{dt}\propto P-N$

   $\therefore\ \dfrac{dN}{dt}=k\,(P-N)$, con $k>0$ una constante.

   (c) $\dfrac{dN}{dt}\propto N\,(P-N)$.

   $\therefore\ \dfrac{dN}{dt}=kN\,(P-N)$, con $k>0$ una constante.

3. (a) $y''=0$

   Cualquier polinomio de primer grado tendrá una segunda derivada igual a cero.

   $\therefore\ y=Ax+B,$ con A,B constantes.

   (b) $y'=3y$

   Proponemos cualquier múltiplo de e^{3x}

   $\therefore\ y=ke^{3x},$ con k constante.

   (c) $xy'+y=3x^2$

   Es suficiente tener una solución potencial de grado 2, para que la suma de ésta con su derivada sea de grado 2.

   $\therefore\ y=kx^2$

   $\therefore\ x(2kx)+\left(kx^2\right)(x)=3x^2$

   $\therefore\ 3kx^2=3x^2 \quad \therefore\ k=1$

   $\therefore\ y=x^2$

   (d) $y'+y=e^x$

   Buscamos una solución que sea múltiplo de e^x. Por lo tanto, proponemos

$y = ke^x$

$\therefore\ y' = ke^x$

$\therefore\ (ke^x) + (ke^x) = e^x$

$\therefore\ 2ke^x = e^x \qquad \therefore\ k = \frac{1}{2}$

$\therefore\ y = \frac{1}{2}e^x$

(e) $y'' + y = 0$

Buscamos una solución que sea el negativo de su segunda derivada. Por ejemplo, proponemos

$y = sen\, x$, o bien, $y = \cos x$

(f) $(y')^2 + y^2 = 1$

La forma de esta ecuación sugiere utilizar $\cos^2 x + sen^2 x = 1$.

$\therefore\ y = sen\, x$, o bien, $y = \cos x$

4. (a) $\dot{x} = 2, \quad x(0) = 2$

$$\frac{dx}{dt} = 2$$

$\therefore\ x(t) = \int 2\, dt = 2t + C.$

Condición inicial:

$x(0) = 2 = 2(0) + C \qquad \therefore\ C = 2$

Solución:

$x(t) = 2t + 2$

(b) $\dot{x} = 5x, \quad x(3) = 2$

$$\frac{dx}{dt} = 5x$$

$$\therefore\ \frac{dx}{x} = 5\, dt$$

$$\therefore\ \int \frac{dx}{x} = 5 \int dt$$ (esto se justifica formalmente en clase)

$\therefore\ \ln|x| = 5t + C$

$\therefore\ |x| = e^{5t+C} = e^{5t}e^C$

$\therefore\ x = \left(\pm e^C\right) e^{5t} = Ae^{5t},$

donde se definió $A = \pm e^C$.

$\therefore\ x(t) = Ae^{5t}$

Condición inicial:

$x(3) = 2 = Ae^{5(3)}$

$\therefore\ A = 2e^{-15}$

$\therefore\ x(t) = (2e^{-15})\, e^{5t}$

Solución:

$x(t) = 2\ e^{5(t-3)}$

(c) $2\dot{x} + x = 0$

$$\frac{dx}{dt} = -\frac{x}{2}$$

Solución:

$$x(t) = Ae^{-t/2}$$

(d) $\dot{x} = 8 - x, \;\; x(0) = 5$

Usamos el teorema: $x(t) = x_h(t) + x_p(t)$

$x_h : \;\; \dot{x}_h = -x_h$

$$\frac{dx_h}{dt} = -x_h$$

$$\therefore x_h(t) = Ae^{-t}$$

$x_p : \;\; \dot{x}_p = 8 - x_p$

Proponemos $x_p(t) = K$

$\therefore \;\; \dot{x}_p(t) = 0$

$\therefore \;\; 0 = 8 - K$

$\therefore \;\; K = 8$

$\therefore \;\; x_p(t) = 8$

$\therefore \;\; x(t) = Ae^{-t} + 8$

Condición inicial:

$x(0) = 5 = A + 8$

$\therefore \;\; A = -3$

Solución:

$$x(t) = -3e^{-t} + 8$$

(e) $\dot{x} = 8 - x, \quad x(0) = 8$

Del inciso anterior, se tiene

$x(t) = Ae^{-t} + 8$

Condición inicial:

$x(0) = 8 = A + 8$

$\therefore A = 0$

Solución:

$$x(t) = 8$$

(f) $\dot{x} - 5x + 10 = 0$

$x_h : \;\; \dot{x}_h - 5x_h = 0$

$\therefore x_h(t) = Ae^{5t}$

$x_p : \;\; \dot{x}_p - 5x_p + 10 = 0$

$\therefore \;\; x_p(t) = 2$

Solución:

$$x(t) = Ae^{5t} + 2$$

5. $\dot{P} = 2\left[D(P) - S(P)\right], S(P) = P - 4, D(P) = 11 - 2P, P(0) = P_0$

$\therefore\ \dot{P} = 2\left[(11 - 2P) - (P - 4)\right]$

$\therefore\ \dot{P} = 30 - 6P$

$P_h:\ \dot{P}_h = -6P_h$

$\therefore\ P_h(t) = Ae^{-6t}$

$P_p:\ \dot{P}_p = 30 - 6P_p$

$\therefore\ P_p(t) = 5$

$\therefore\ P(t) = Ae^{-6t} + 5$

Condición inicial:

$P(0) = P_0 = A + 5$

$\therefore\ A = P_0 - 5$

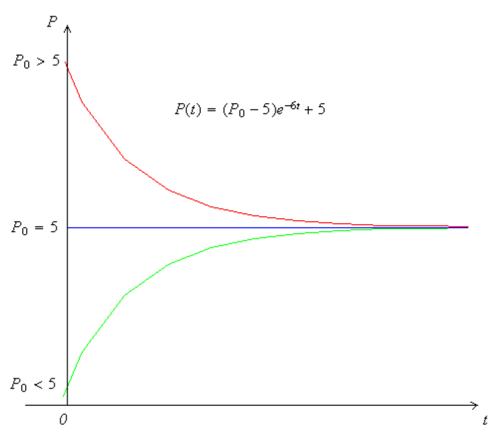
Solución:

$P(t) = (P_0 - 5)\, e^{-6t} + 5$

Comportamiento a largo plazo:

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \left[(P_0 - 5)\, e^{-6t} + 5\right] = 5$$

Esto significa que el precio tiene a estabilizarse en 5.



6. $\dot{P} = \lambda\left[D(P) - S(P)\right],\ S(P) = \alpha + \beta P,\ D(P) = a - bP,$

$a, b, \alpha, \beta > 0,\ a > \alpha$

$\dot{P} = \lambda[(a - bP) - (\alpha + \beta P)]$

$\therefore \dot{P} + \lambda\,(b + \beta)\, P = \lambda\,(a - \alpha)$

$P_h:\ \dot{P}_h = -\lambda\,(b + \beta)\, P_h$

$\therefore\ P_h(t) = Ae^{-\lambda(b+\beta)t}$

$P_p: \ \dot{P}_p + \lambda(b+\beta) P_p = \lambda(a-\alpha)$

$\therefore \ P_p(t) = \dfrac{a-\alpha}{b+\beta}$

Solución:

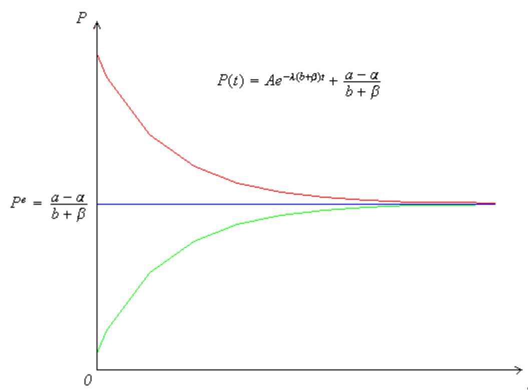
$P(t) = Ae^{-\lambda(b+\beta)t} + \dfrac{a-\alpha}{b+\beta}$

Comportamiento a largo plazo:

Como $\lambda, b, \beta > 0$, por lo tanto

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \left[Ae^{-\lambda(b+\beta)t} + \frac{a-\alpha}{b+\beta} \right] = \frac{a-\alpha}{b+\beta} = P^e$$

Por lo tanto, el precio tiene a un precio de equilibrio $P^e = \dfrac{a-\alpha}{b+\beta}$.



7. (a) $\dfrac{dP}{dt} = \alpha P, \ \ \alpha > 0, \ \ P(0) = P_0$

   $\therefore \ P(t) = Ae^{\alpha t}$

   Condición inicial:

   $P(0) = P_0 = A$

   Solución:

   $P(t) = P_0 e^{\alpha t}$

   (b) Se busca $t = t^*$ tal que $P(t^*) = 2P_0$

   $\therefore \ P(t^*) = 2P_0 = P_0 e^{\alpha t^*}$

   $\therefore \ e^{\alpha t^*} = 2$

   $\therefore \ \alpha t^* = \ln 2$

   $\therefore \ t^* = \dfrac{\ln 2}{\alpha}$ (independientemente del valor de P_0)

   (c) Como $\alpha > 0$, por lo tanto

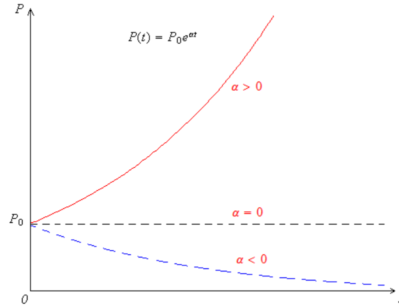
   $$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} (P_0 e^{\alpha t}) = \infty$$

   Esto significa que la población crece indefinidamente.

(d) Si $\alpha < 0$, entonces $\alpha = -|\alpha| < 0$. En ese caso,

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \left(P_0 e^{-|\alpha|t}\right) = 0$$

Esto implicaría que la población se extingue a la larga.



8. $\dfrac{dP}{dt} = (\alpha - \beta)\,P, \;\; \alpha, \beta > 0, \;\; P(0) = P_0$

$\therefore \;\; P(t) = Ae^{(\alpha-\beta)t}$

Condición inicial:

$P(0) = P_0 = A$

Solución:

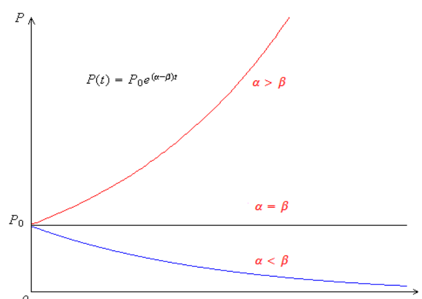
$P(t) = P_0 e^{(\alpha-\beta)t}$

Casos:

i. $\alpha > \beta \;\Rightarrow\; \alpha - \beta > 0 \;\Rightarrow\; \lim_{t\to\infty} P(t) = \lim_{t\to\infty} \left[P_0 e^{(\alpha-\beta)t}\right] = \infty.$
Por lo tanto, si la tasa de nacimientos es mayor que la de muertes, la población crece sin límites.

ii. $\alpha = \beta \;\Rightarrow\; \alpha - \beta = 0 \;\Rightarrow\; \lim_{t\to\infty} P(t) = \lim_{t\to\infty} P_0 = P_0.$
Por lo tanto, si las dos tasas son iguales, la población se mantiene constante.

iii. $\alpha < \beta \;\Rightarrow\; \alpha - \beta < 0 \;\Rightarrow\; \lim_{t\to\infty} P(t) = \lim_{t\to\infty} \left[P_0 e^{(\alpha-\beta)t}\right] = 0.$
Por lo tanto, si la tasa de nacimientos es menor que la de muertes, la poblaciòn a largo plazo desaparece.



9. $\dfrac{dP}{dt} = \alpha P - E, \quad \alpha, E > 0,\ P(0) = P_0$

$P_h:\ \dot{P}_h = \alpha P_h$

$\quad \therefore\ P_h(t) = Ae^{\alpha t}$

$P_p:\ \dot{P}_p = \alpha P_p - E$

$\quad \therefore\ P_p(t) = \dfrac{E}{\alpha}$

$\therefore\ P(t) = Ae^{\alpha t} + \dfrac{E}{\alpha}$

Condición inicial:

$$P(0) = P_0 = A + \frac{E}{\alpha} \qquad \therefore\ A = P_0 - \frac{E}{\alpha}$$

Solución:

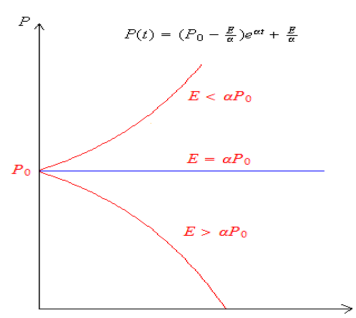
$$P(t) = \left(P_0 - \frac{E}{\alpha}\right) e^{\alpha t} + \frac{E}{\alpha}$$

Casos:

i. $\dfrac{E}{\alpha} > P_0 \ \Rightarrow\ A < 0 \ \Rightarrow\ \lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty}\left[Ae^{\alpha t} + \dfrac{E}{\alpha}\right] = -\infty.$
Por lo tanto, emigran más de los que nacen, la población se extingue.

ii. $\dfrac{E}{\alpha} = P_0 \ \Rightarrow\ A = 0 \ \Rightarrow\ \lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty}\left[Ae^{\alpha t} + \dfrac{E}{\alpha}\right] = \dfrac{E}{\alpha} = P_0.$
Por lo tanto, si emigran el mismo número que los que nacen, la población se mantiene constante.

iii. $\dfrac{E}{\alpha} < P_0 \ \Rightarrow\ A > 0 \ \Rightarrow\ \lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty}\left[Ae^{\alpha t} + \dfrac{E}{\alpha}\right] = \infty.$
Por lo tanto, si emigran menos que los que nacen, la población crece sin límite.



10. Partimos del sistema

$$C(t) + I(t) = Y(t) \qquad (1)$$
$$I(t) = k\dot{C}(t) \qquad (2)$$
$$C(t) = aY(t) + b, \qquad (3)$$

con $a, b, k \in \mathbb{R}^+$, $a < 1$. Sustituimos (3) en (1), y de esta última despejamos I, obteniendo

$$I(t) = (1-a)Y(t) - b. \qquad (4)$$

Sustituimos $I(t)$ de (4) en (2), obteniendo

$$\dot{C}(t) = \frac{1}{k}\left[(1-a)Y(t) - b\right]. \quad (5)$$

Por otra parte, tomando la derivada de la ecuación (3) con respecto a t se obtiene

$$\dot{C}(t) = a\dot{Y}(t). \qquad (6)$$

Por último, igualamos (5) y (6) para eliminar $\dot{C}(t)$, de donde

$$\dot{Y} = \left(\frac{1-a}{ka}\right)Y - \frac{b}{ka}. \qquad (7)$$

Así, se trata de resolver $\dot{Y} = \left(\frac{1-a}{ka}\right)Y - \frac{b}{ka}$, $b, k > 0$, $0 < a < 1$, $Y(0) = Y_0$.

$Y_h:\ \dot{Y}_h = \left(\frac{1-a}{ka}\right)Y_h$

$\therefore\ Y_h(t) = Ae^{\left(\frac{1-a}{ka}\right)t}$

$Y_p:\ \dot{Y}_p = \left(\frac{1-a}{ka}\right)Y_p - \frac{b}{ka}$

$\therefore\ Y_p(t) = \frac{b}{1-a}$

$\therefore\ Y(t) = Ae^{\left(\frac{1-a}{ka}\right)t} + \frac{b}{1-a}$

Condición inicial:

$$Y(0) = Y_0 = A + \frac{b}{1-a} \quad \therefore\ A = Y_0 - \frac{b}{1-a} > 0$$

Solución:

$$Y(t) = \left(Y_0 - \frac{b}{1-a}\right) e^{\left(\frac{1-a}{ka}\right)t} + \frac{b}{1-a}$$

Función $I(t)$:

Sustituyendo $Y(t)$ en la ecuación (4), obtenemos

$$I(t) = (1-a)Y(t) - b = (1-a)\left[\left(Y_0 - \frac{b}{1-a}\right) e^{\left(\frac{1-a}{ka}\right)t} + \frac{b}{1-a}\right] - b$$

$$\therefore \; I(t) = (1-a)\left(Y_0 - \frac{b}{1-a}\right) e^{\left(\frac{1-a}{ka}\right)t}$$

Comportamiento asintótico de $\dfrac{Y(t)}{I(t)}$:

$$\lim_{t\to\infty} \frac{Y(t)}{I(t)} = \lim_{t\to\infty} \frac{\left(Y_0 - \frac{b}{1-a}\right) e^{\left(\frac{1-a}{ka}\right)t} + \frac{b}{1-a}}{(1-a)\left(Y_0 - \frac{b}{1-a}\right) e^{\left(\frac{1-a}{ka}\right)t}}$$

$$\therefore \; \lim_{t\to\infty} \frac{Y(t)}{I(t)} = \lim_{t\to\infty} \left[\frac{1}{1-a} + \frac{b}{(1-a)^2\left(Y_0 - \frac{b}{1-a}\right) e^{\left(\frac{1-a}{ka}\right)t}}\right]$$

$$\therefore \; \lim_{t\to\infty} \frac{Y(t)}{I(t)} = \frac{1}{1-a} + \frac{b}{(1-a)^2\left(Y_0 - \frac{b}{1-a}\right) \lim_{t\to\infty} e^{\left(\frac{1-a}{ka}\right)t}}$$

$$\therefore \; \lim_{t\to\infty} \frac{Y(t)}{I(t)} = \frac{1}{1-a},$$

en donde se usó que $\lim_{t\to\infty} e^{\left(\frac{1-a}{ka}\right)t} = \infty$, ya que $\alpha < 1$.

11. (a) $\dot{x} = -2t, \quad x(1) = 3$

$$x(t) = \int \; 2t \; dt = -t^2 + C$$

Condición inicial:

$$x(1) = 3 = -1 + C \qquad \therefore \; C = 4$$

Solución:

$$x(t) = -t^2 + 4$$

(b) $\dot{x} + (2\cos t)x = \cos t$

Usamos el método del factor de integración $\mu(t)$:

$$\mu(t) = e^{\int 2\cos t \, dt} = e^{2\,sen\,t}$$

$$e^{2\,sen\,t}\left[\dot{x} + (2\cos t)x\right] = e^{2\,sen\,t}\cos t$$

$$\therefore \; \frac{d}{dt}\left[e^{2\,sen\,t}x(t)\right] = e^{2\,sen\,t}\cos t$$

$\therefore\ e^{2\,sen\,t}x(t) = \int e^{2\,sen\,t}\cos t\ dt = \dfrac{1}{2}e^{2\,sen\,t} + C$

Solución:

$x(t) = \dfrac{1}{2} + Ce^{-2\,sen\,t}$

(c) $\dot{x} - 2tx = t(1+t^2)$

$\mu(t) = e^{\int -2t\ dt} = e^{-t^2}$

$e^{-t^2}\left[\dot{x} - 2tx\right] = t(1+t^2)e^{-t^2}$

$\therefore\ \dfrac{d}{dt}\left[e^{-t^2}x(t)\right] = t(1+t^2)e^{-t^2}$

$\therefore\ e^{-t^2}x(t) = \int (t+t^3)e^{-t^2}dt$

$\therefore\ e^{-t^2}x(t) = \int te^{-t^2}dt + \int t^3e^{-t^2}dt + C$

Integramos por partes el segundo término:

$\int t^3e^{-t^2}dt = \int \underbrace{t^2}_{u}\underbrace{te^{-t^2}}_{dv}dt = -\dfrac{t^2}{2}e^{-t^2} + \int te^{-t^2}dt$

$\therefore\ e^{-t^2}x(t) = \int te^{-t^2}dt + \left(-\dfrac{t^2}{2}e^{-t^2} + \int te^{-t^2}dt\right) + C$

$\therefore\ e^{-t^2}x(t) = 2\int te^{-t^2}dt - \dfrac{t^2}{2}e^{-t^2} + C$

Integramos por sustitución el primer término:

$2\int te^{-t^2}dt = -e^{-t^2}$

$\therefore\ e^{-t^2}x(t) = -e^{-t^2} - \dfrac{t^2}{2}e^{-t^2} + C$

Solución:

$x(t) = -\left(1 + \dfrac{t^2}{2}\right) + Ce^{t^2}$

(d) $2\dot{x} + 12x + 2e^t = 1\ \therefore\ \dot{x} + 6x = \dfrac{1}{2} - e^t$

$\mu(t) = e^{\int 6dt} = e^{6t}$

$e^{6t}\left[\dot{x} + 6x\right] = \left[\dfrac{1}{2} - e^t\right]e^{6t}$

$\therefore\ \dfrac{d}{dt}\left[e^{6t}x(t)\right] = \dfrac{1}{2}e^{6t} - e^{7t}$

$\therefore\ e^{6t}x(t) = \int \left(\dfrac{1}{2}e^{6t} - e^{7t}\right)dt = \dfrac{1}{12}e^{6t} - \dfrac{1}{7}e^{7t} + C$

Solución:

$x(t) = \dfrac{1}{12} - \dfrac{1}{7}e^t + Ce^{-6t}$

(e) $\dot{x} + t^2x = 5t^2, \quad x(0) = 6$

$\mu(t) = e^{\int t^2dt} = e^{t^3/3}$

$$e^{t^3/3}\left[\dot{x}+t^2x\right]=5t^2e^{t^3/3}$$

$$\therefore\ \frac{d}{dt}\left[e^{t^3/3}x(t)\right]=5t^2e^{t^3/3}$$

$$\therefore\ e^{t^3/3}x(t)=\int 5t^2e^{t^3/3}dt=5e^{t^3/3}+C$$

$$\therefore\ x(t)=5+Ce^{-t^3/3}$$

Condición inicial:

$$x(0)=6=5+C\ \therefore\ C=1$$

Solución:

$$x(t)=5+e^{-t^3/3}$$

(f) $\dot{x}=-\dfrac{2x}{t}+\dfrac{1}{t^3},\ \ x(1)=3$

$$\dot{x}+\frac{2}{t}x=\frac{1}{t^3}$$

$$\mu(t)=e^{\int\frac{2}{t}dt}=e^{2\ln|t|}=e^{\ln|t|^2}=e^{\ln t^2}=t^2$$

$$t^2\left[\dot{x}+\frac{2}{t}x\right]=t^2t^{-3}$$

$$\therefore\ \frac{d}{dt}\left[t^2x(t)\right]=\frac{1}{t}$$

$$\therefore\ t^2x(t)=\int\frac{1}{t}dt=\ln|t|+C$$

$$\therefore\ x(t)=\frac{\ln|t|+C}{t^2}$$

Condición inicial:

$$x(1)=3=\frac{\ln|1|+C}{1}\ \therefore\ C=3$$

Solución:

$$\therefore\ x(t)=\frac{3+\ln|t|}{t^2}$$

(g) $t\dot{x}+2x=t^{-3},\quad x(1)=2$

$$\dot{x}+\frac{2}{t}x=t^{-4}$$

$$\mu(t)=e^{\int\frac{2}{t}dt}=e^{2\ln|t|}=e^{\ln|t|^2}=e^{\ln t^2}=t^2$$

$$t^2\left[\dot{x}+\frac{2}{t}x\right]=t^2t^{-4}$$

$$\therefore\ \frac{d}{dt}\left[t^2x(t)\right]=\frac{1}{t^2}$$

$$\therefore\ t^2x(t)=\int\frac{1}{t^2}dt=-\frac{1}{t}+C$$

$$\therefore\ x(t)=-\frac{1}{t^3}+\frac{C}{t^2}$$

Condición inicial:

$$x(1) = 2 = -\frac{1}{1} + \frac{C}{1} \quad \therefore \quad C = 3$$

Solución:

$$x(t) = -\frac{1}{t^3} + \frac{3}{t^2}$$

12. (a) $\dot{y} - 2ty = e^{t^2}$

$$\mu(t) = e^{\int(-2t)dx} = e^{-t^2}$$

$$e^{-t^2}\left[\dot{y} - 2ty\right] = e^{-t^2}e^{t^2}$$

$$\therefore \quad \frac{d}{dt}\left[e^{-t^2}y(t)\right] = 1$$

$$\therefore \quad e^{-t^2}y(t) = \int dt$$

$$\therefore \quad e^{-t^2}y(t) = t + C$$

Solución:

$$y(t) = (t + C)\, e^{t^2}$$

(b) $\lambda' = -\alpha^2\lambda + 5\alpha^2$

$$\lambda' + \alpha^2\lambda = 5\alpha^2$$

$$\mu(\alpha) = e^{\int \alpha^2 d\alpha} = e^{\alpha^3/3}$$

$$e^{\alpha^3/3}\left[\lambda' + \alpha^2\lambda\right] = 5\alpha^2 e^{\alpha^3/3}$$

$$\therefore \quad \frac{d}{d\alpha}\left[e^{\alpha^3/3}\lambda(\alpha)\right] = 5\alpha^2 e^{\alpha^3/3}$$

$$\therefore \quad e^{\alpha^3/3}\lambda(\alpha) = \int 5\alpha^2 e^{\alpha^3/3}dx$$

$$\therefore \quad e^{\alpha^3/3}\lambda(\alpha) = 5e^{\alpha^3/3} + C$$

Solución:

$$\lambda(\alpha) = 5 + Ce^{-\alpha^3/3}$$

(c) $x' + (2\cos\theta)x = \cos\theta$

$$\mu(\theta) = e^{\int 2\cos\theta \, d\theta} = e^{2\,sen\,\theta}$$

$$e^{2\,sen\,\theta}\left[x' + (2\cos\theta)x\right] = e^{2\,sen\,\theta}\cos\theta$$

$$\therefore \quad \frac{d}{d\theta}\left[e^{2\,sen\,\theta}x\,(\theta)\right] = (\cos\theta)\,e^{2\,sen\,\theta}$$

$$\therefore \quad e^{2\,sen\,\theta}x\,(\theta) = \int(\cos\theta)\,e^{2sen\theta}d\theta$$

$$\therefore \quad e^{2\,sen\,\theta}x\,(\theta) = \frac{1}{2}e^{2\,sen\,\theta} + C$$

Solución:

$$x(\theta) = \frac{1}{2} + Ce^{-2\,sen\,\theta}$$

(d) $\dfrac{dy}{dx} = x + \dfrac{y}{2}$

$$\frac{dy}{dx} - \frac{y}{2} = x$$

$\mu(x) = e^{\int -\frac{1}{2}dx} = e^{-x/2}$

$\therefore\ \dfrac{d}{dx}\left[e^{-x/2}y(x)\right] = xe^{-x/2}$

$\therefore\ e^{-x/2}y(x) = \int xe^{-x/2}dx$

$\therefore\ e^{-x/2}y(x) = -2xe^{-x/2} - 4e^{-x/2} + C$

Solución:

$y(x) = -2x - 4 + Ce^{x/2}$

(e) $y' + 3u^2y = u^2, \quad y(0) = 1$

$\mu(u) = e^{\int 3u^2du} = e^{u^3}$

$e^{u^3}\left[y' + 3u^2y\right] = u^2e^{u^3}$

$\therefore\ \dfrac{d}{du}\left[e^{u^3}y(u)\right] = e^{u^3}u^2$

$\therefore\ e^{u^3}y(u) = \int e^{u^3}u^2du$

$\therefore\ e^{u^3}y(u) = \dfrac{1}{3}e^{u^3} + C$

$\therefore\ y(u) = \dfrac{1}{3} + Ce^{-u^3}$

Condición inicial:

$y(0) = 1 = \dfrac{1}{3} + C\ \ \therefore\ \ C = \dfrac{2}{3}$

Solución:

$y(u) = \dfrac{1}{3} + \dfrac{2}{3}e^{-u^3}$

(f) $\dfrac{dx}{dy} + 2x = e^y, \quad x(0) = 1$

$\mu(y) = e^{\int 2dy} = e^{2y}$

$e^{2y}\left[\dfrac{dx}{dy} + 2x\right] = e^{2y}e^y$

$\therefore\ \dfrac{d}{dy}\left[e^{2y}x(y)\right] = e^{3y}$

$\therefore\ e^{2y}x(y) = \int e^{3y}dy$

$\therefore\ e^{2y}x(y) = \dfrac{1}{3}e^{3y} + C$

$\therefore\ x(y) = \dfrac{1}{3}e^y + Ce^{-2y}$

Condición inicial:

$x(0) = 1 = \dfrac{1}{3} + C\ \ \therefore\ \ C = \dfrac{2}{3}$

Solución:

$x(y) = \dfrac{1}{3}e^y + \dfrac{2}{3}e^{-2y}$

(g) $xy' + 5y = 7x^2, \quad y(2) = 5$

$$y' + \frac{5}{x}y = 7x$$

$$\mu(x) = e^{\int \frac{5}{x}dx} = e^{5\ln|x|} = |x|^5$$

$$x > 0 \Longrightarrow x^5 \left[y' + \frac{5}{x}y\right] = (7x)\, x^5$$

$$x < 0 \Longrightarrow (-x^5) \left[y' + \frac{5}{x}y\right] = (7x)\,(-x^5) \quad \text{(se cancela el } -\text{)}$$

En cualquier caso, se obtiene

$$x^5 \left[y' + \frac{5}{x}y\right] = (7x)\, x^5$$

$$\therefore \ \frac{d}{dx}\left[x^5\, y(x)\right] = 7x^6$$

$$\therefore \ x^5 y(x) = \int 7x^6\, dx$$

$$\therefore \ x^5 y(x) = x^7 + C$$

$$\therefore \ y(x) = x^2 + \frac{C}{x^5}$$

Condiciòn inicial:

$$y(2) = 5 = 2^2 + \frac{C}{2^5} \ \therefore \ C = 32$$

Solución:

$$y(x) = x^2 + \frac{32}{x^5}$$

(h) $(t^2 + 4)\dot{y} + 3ty = t, \quad y(0) = 1$

$$\dot{y} + \frac{3t}{(t^2+4)}y = \frac{t}{(t^2+4)}$$

$$\mu(t) = e^{\int \frac{3t}{(t^2+4)}dt} = e^{\frac{3}{2}\ln\left(t^2+4\right)} = e^{\ln\left(t^2+4\right)^{3/2}} = \left(t^2+4\right)^{3/2}$$

$$\left(t^2+4\right)^{3/2}\left[\dot{y} + \frac{3t}{(t^2+4)}y\right] = \frac{t}{(t^2+4)}\left(t^2+4\right)^{3/2}$$

$$\therefore \ \frac{d}{dt}\left[\left(t^2+4\right)^{3/2} y(t)\right] = t\,\left(t^2+4\right)^{1/2}$$

$$\therefore \ \left(t^2+4\right)^{3/2} y(t) = \int t\,\left(t^2+4\right)^{1/2} dt$$

$$\therefore \ \left(t^2+4\right)^{3/2} y(t) = \frac{1}{3}\left(t^2+4\right)^{3/2} + C$$

$$\therefore \ y(t) = \frac{1}{3} + \frac{C}{\left(t^2+4\right)^{3/2}}$$

Condiciòn inicial:

$$y(0) = 1 = \frac{1}{3} + \frac{C}{4^{3/2}} \ \therefore \ C = \frac{16}{3}$$

Solución:

$$y(t) = \frac{1}{3} + \frac{16}{3\left(t^2+4\right)^{3/2}}$$

(i) $\dot{x} = x - t, \quad x(0) = x_0$

$\dot{x} - x = -t$

$\mu(t) = e^{\int(-1)dt} = e^{-t}$

$e^{-t}\left[\dot{x} - x\right] = -te^{-t}$

$\therefore \ \frac{d}{dt}\left[e^{-t}x(t)\right] = -te^{-t}$

$\therefore \ e^{-t}x(t) = \int -te^{-t}dt$

$\therefore \ e^{-t}x(t) = te^{-t} + e^{-t} + C$

$\therefore \ x(t) = t + 1 + Ce^{t}$

Condiciòn inicial:

$x(0) = x_0 = 1 + C \ \therefore \ C = x_0 - 1$

Solución:

$x(t) = t + 1 + (x_0 - 1)\, e^{t}$

13. Partimos del sistema ($\sigma, \alpha, H_0, \mu \in \mathbb{R}^+$, $\alpha\sigma \neq \mu$):

$$X(t) = \sigma K(t) \quad (1)$$
$$\dot{K}(t) = \alpha X(t) + H(t) \quad (2)$$
$$H(t) = H_0 e^{\mu t}, \quad (3)$$

Sustituimos (1) y (3) en (2), obteniendo

$$\dot{K}(t) = \alpha\sigma K(t) + H_0 e^{\mu t}. \quad (4)$$

Así, se trata de resolver $\dot{K} - (\alpha\sigma)\, K = H_0 e^{\mu t}$, $\sigma, \alpha, H_0, \mu \in \mathbb{R}^+$, $\alpha\sigma \neq \mu$, $K(0) = K_0$.

Factor de integración$= e^{\int -(\alpha\sigma)dt} = e^{-\alpha\sigma t}$.

$\therefore \ e^{-\alpha\sigma t}\left[\dot{K} - (\alpha\sigma)\, K\right] = e^{-\alpha\sigma t}\left[H_0 e^{\mu t}\right]$

$\therefore \ \frac{d}{dt}\left[e^{-\alpha\sigma t}K(t)\right] = H_0 e^{(\mu - \alpha\sigma)t}$

$\therefore \ e^{-\alpha\sigma t}K(t) = \int H_0 e^{(\mu - \alpha\sigma)t}dt$

$\therefore \ e^{-\alpha\sigma t}K(t) = \frac{H_0}{\mu - \alpha\sigma}e^{(\mu - \alpha\sigma)t} + C$

$\therefore \ K(t) = \frac{H_0}{\mu - \alpha\sigma}e^{\mu t} + Ce^{\alpha\sigma t}$

Condición inicial:

$K(0) = K_0 = \frac{H_0}{\mu - \alpha\sigma} + C \ \therefore \ C = K_0 - \frac{H_0}{\mu - \alpha\sigma}$

Solución:

$K(t) = \frac{H_0}{\mu - \alpha\sigma}e^{\mu t} + \left(K_0 - \frac{H_0}{\mu - \alpha\sigma}\right)e^{\alpha\sigma t}$

14. (a) $y'(x) = e^{x^2}, \;\; y(2) = 5; \qquad y(x) = 5 + \int\limits_2^x e^{s^2} ds$

Sea $y(x) = 5 + \int\limits_2^x e^{s^2} ds$. Por una parte,

$$y'(x) = \frac{d}{dx}\left(5 + \int\limits_2^x e^{s^2} ds\right) = \frac{d}{dx}\left(\int\limits_2^x e^{s^2} ds\right) = e^{x^2}$$

Por otra parte,

$$y(2) = 5 + \int\limits_2^2 e^{s^2} ds = 5$$

(b) $y'(x) + 2xy(x) = 1, \;\; y(2) = 0; \qquad y(x) = e^{-x^2} \int\limits_2^x e^{s^2} ds$

Sea $y(x) = e^{-x^2} \int\limits_2^x e^{s^2} ds$. Por una parte,

$$y'(x) = \frac{d}{dx}\left(e^{-x^2} \int\limits_2^x e^{s^2} ds\right) = \frac{d}{dx}\left(e^{-x^2}\right) \int\limits_2^x e^{s^2} ds + e^{-x^2} \frac{d}{dx}\left(\int\limits_2^x e^{s^2} ds\right)$$

$$\therefore \;\; y'(x) = \left(-2x \; e^{-x^2}\right) \int\limits_2^x e^{s^2} ds + e^{-x^2}\left(e^{x^2}\right)$$

$$\therefore \;\; y'(x) = -2x \;\left(e^{-x^2} \int\limits_2^x e^{s^2} ds\right) + 1$$

$$\therefore \;\; y'(x) = -2x \; y(x) + 1$$

$$\therefore \;\; y'(x) + 2xy(x) = 1$$

Por otra parte,

$$y(2) = e^{-2^2} \int\limits_2^2 e^{s^2} ds = 0$$

(c) $\dot{B}(t) = r(t)B(t), \;\; B(0) = B_0; \qquad B(t) = B_0 \; e^{\int_0^t r(s)ds}$

Sea $B(t) = B_0 \; e^{\int_0^t r(s)ds}$. Por una parte,

$$\dot{B}(t) = B_0 \; e^{\int_0^t r(s)ds}\left(\frac{d}{dt}\int\limits_0^t r(s)ds\right)$$

$$\therefore \;\; \dot{B}(t) = \left[B_0 \; e^{\int_0^t r(s)ds}\right] r(t)$$

$$\therefore \;\; \dot{B}(t) = B(t)r(t)$$

Por otra parte,

$$B(0) = B_0 \; e^{\int_0^0 r(s)ds} = B_0$$

15. $\dfrac{dx}{dt} + \dfrac{2x}{t} = \dfrac{sen\, t}{t^3}, \;\; x(2) = 3, \;\; t \geq 2$

$$\dot{x} + \frac{2x}{t} = \frac{sen\, t}{t^3}$$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2\left[\dot{x}+\frac{2x}{t}\right]=t^2\left[\frac{sen\,t}{t^3}\right]$$

$$\therefore\ \frac{d}{dt}\left[t^2x(t)\right]=\frac{sent}{t}$$

$$\therefore\ t^2x(t)=\int\frac{sent}{t}dt$$

La integral del lado derecho no posee una antiderivada simple. Por lo tanto, debemos utilizar integrales definidas con límite variable (Teorema Fundamental del Cálculo), de donde

$$t^2x(t)=\int_a^t\frac{sen\,s}{s}ds.$$

Sabemos que $x(2)=3$, por lo que sustituimos $t=2$ en la integral anterior, esto es,

$$2^2x(2)=\int_a^2\frac{sen\,s}{s}ds,$$

$$\therefore\ 12=\int_a^2\frac{sen\,s}{s}ds$$

De esta manera,

$$t^2x(t)-12=\int_a^t\frac{sen\,s}{s}ds-\int_a^2\frac{sen\,s}{s}ds$$

$$\therefore\ t^2x(t)-12=\int_a^t\frac{sen\,s}{s}ds+\int_2^a\frac{sen\,s}{s}ds$$

$$\therefore\ t^2x(t)-12=\int_2^t\frac{sen\,s}{s}ds.$$

$$\therefore\ t^2x(t)=12+\int_2^t\frac{sen\,s}{s}ds$$

Solución:

$$x(t)=\frac{1}{t^2}\left[12+\int_2^t\frac{sen\,s}{s}ds\right].$$

16. $\dfrac{dy}{dx}-\dfrac{y}{x}=G(x), \quad y(3)=6, \quad x\geq 3.$

$$\frac{dy}{dx}-\frac{1}{x}y=G(x)$$

$$\mu(x)=e^{\int\left(-\frac{1}{x}\right)dx}=e^{-\ln x}=\frac{1}{e^{\ln x}}=\frac{1}{x}$$

$$\frac{1}{x}\left[\frac{dy}{dx}-\frac{1}{x}y\right]=\frac{G(x)}{x}$$

$$\therefore\ \frac{d}{dx}\left[\frac{1}{x}y\left(x\right)\right]=\frac{G(x)}{x}$$

$$\therefore\ \frac{1}{x}y\left(x\right)=\int_a^x\frac{G(u)}{u}du$$

$$\therefore\ \frac{1}{3}y\left(3\right)=\int_a^3\frac{G(u)}{u}du$$

$$\therefore\ \frac{1}{x}y\left(x\right)-\frac{1}{3}y\left(3\right)=\int_a^x\frac{G(u)}{u}du-\int_a^3\frac{G(u)}{u}du$$

$$\therefore\ \frac{1}{x}y\left(x\right)-\frac{1}{3}\left(6\right)=\int_3^x\frac{G(u)}{u}du$$

Solución:

$$\therefore\ y(x)=x\left[2+\int_3^x\frac{G(u)}{u}du\right]$$

17. $\dfrac{dy}{dx}=1+2xy, \ \ y(0)=3$

$$\frac{dy}{dx}-2xy=1$$

$$\mu(x)=e^{\int(-2x)dx}=e^{-x^2}$$

$$e^{-x^2}\left[\frac{dy}{dx}-2xy\right]=e^{-x^2}$$

$$\therefore\ \frac{d}{dx}\left[e^{-x^2}y(x)\right]=e^{-x^2}$$

$$\therefore\ e^{-x^2}y(x)=\int_a^x e^{-t^2}dt$$

$$\therefore\ e^{0}y(0)=\int_a^0 e^{-t^2}dt$$

$$\therefore\ e^{-x^2}y(x)-y(0)=\int_a^x e^{-t^2}dt-\int_a^0 e^{-t^2}dt$$

$$\therefore\ e^{-x^2}y(x)-y(0)=\int_0^x e^{-t^2}dt$$

$$\therefore\ e^{-x^2}y(x)-3=\int_0^x e^{-t^2}dt$$

$$\therefore\ y(x)=e^{x^2}\left[3+\int_0^x e^{-t^2}dt\right]=e^{x^2}\left[3+\frac{\sqrt{\pi}}{2}\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\right]$$

Solución:

$$\therefore\ y(x)=e^{x^2}\left[3+\frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]$$

18. $\dot{p}=\frac{1}{\lambda}p-\frac{1}{\lambda}m(t),\quad p(t_0)=p_0,\quad t\geq t_0.$

$$\dot{p}-\frac{1}{\lambda}p=-\frac{1}{\lambda}m(t)$$

$$\mu(t)=e^{\int\left(-\frac{1}{\lambda}\right)dt}=e^{-t/\lambda}$$

$$e^{-t/\lambda}\left[\dot{p}-\frac{1}{\lambda}p\right]=e^{-t/\lambda}\left[-\frac{1}{\lambda}m(t)\right]$$

$$\therefore\ \frac{d}{dt}\left[e^{-t/\lambda}p(t)\right]=-\frac{1}{\lambda}e^{-t/\lambda}m(t)$$

$$\therefore\ e^{-t/\lambda}p(t)=-\frac{1}{\lambda}\int_a^t e^{-s/\lambda}m(s)\,ds$$

$$\therefore\ e^{-t_0/\lambda}p_0=-\frac{1}{\lambda}\int_a^{t_0} e^{-s/\lambda}m(s)\,ds$$

$$\therefore\ e^{-t/\lambda}p(t)-e^{-t_0/\lambda}p_0=-\frac{1}{\lambda}\int_{t_0}^t e^{-s/\lambda}m(s)\,ds$$

Solución:

$$p(t) = e^{t/\lambda}\left[e^{-t_0/\lambda}p_0 - \frac{1}{\lambda}\int_{t_0}^{t} e^{-s/\lambda}m(s)\,ds\right]$$

o bien,

$$p(t) = e^{(t-t_0)/\lambda}p_0 - \frac{1}{\lambda}\int_{t_0}^{t} e^{(t-s)/\lambda}m(s)\,ds$$

19. $\dot{Y} = rY - X(t), \quad Y(T) = Y_T, \;\; r > 0,\; 0 \leq t \leq T.$

(a) $\dot{Y} - rY = -X(t)$

$\therefore\ \mu(t) = e^{\int_T^t (-r)dt} = e^{-r(t-T)}$

(b) $e^{-r(t-T)}\left[\dot{Y} - rY\right] = -X(t)e^{-r(t-T)}$

$$\frac{d}{dt}\left[e^{-r(t-T)}Y(t)\right] = -e^{-r(t-T)}X(t)$$

$$\therefore\ e^{-r(t-T)}Y(t) = -\int_a^t e^{-r(s-T)}X(s)\,ds$$

$$\therefore\ e^{0}Y(T) = -\int_a^T e^{-r(s-T)}X(s)\,ds$$

$$\therefore\ e^{-r(t-T)}Y(t) - Y(T) = -\int_a^t e^{-r(s-T)}X(s)\,ds + \int_a^T e^{-r(s-T)}X(s)\,ds$$

$$\therefore\ e^{-r(t-T)}Y(t) - Y_T = \int_t^T e^{-r(s-T)}X(s)\,ds$$

$$\therefore\ Y(t) = e^{r(t-T)}\left[Y_T + \int_t^T e^{-r(s-T)}X(s)\,ds\right]$$

$$\therefore\ Y(t) = e^{r(t-T)}Y_T + \int_t^T e^{r(t-T)}e^{-r(s-T)}X(s)\,ds$$

$$\therefore\ Y(t) = e^{r(t-T)}Y_T + \int_t^T e^{-r(s-t)}X(s)\,ds$$

(c) $$Y(t)=\lim_{T\to\infty}\left[e^{r(t-T)}Y_T\right]+\lim_{T\to\infty}\int_t^T e^{-r(s-t)}X(s)\,ds$$

Como $r>0$, por tanto $\lim_{T\to\infty} e^{r(t-T)}=0$, de donde

$$Y(t)=\int_t^\infty e^{-r(s-t)}X(s)\,ds.$$

(d) Sea $\tau(s)=s-t$. Así, $d\tau=ds$. Los nuevos límites de integración son $\tau(t)=t-t=0$ y $\tau(\infty)=\infty-t=\infty$. Así,

$$Y(t)=\int_t^\infty e^{-r(s-t)}X(s)\,ds=\int_0^\infty e^{-r\tau}X(\tau+t)\,d\tau.$$

El valor de la inversión al tiempo t es la suma de los flujos de inversión a tiempos posteriores, $X(\tau+t)$, descontados al tiempo τ, donde $0\leq\tau<\infty$.

20. Modelo general (visto en clase):

$$\dot{Y}=r(t)Y+\delta(t)B(t),\text{...}(1)$$

con

$$B(t)=B_T\ e^{-\int_t^T\ r(s)ds},\text{ para todo } 0\leq t\leq T\text{......................}(2)$$

Modelo particular:

$$\dot{Y}=\left(r_0-\frac{1}{t+1}\right)Y-\frac{T+1}{t+1},\text{...}(3)$$

$Y(T)=1$, $r_0>0$ constante.

(a) Comparando las ecuaciones (1) y (3) se observa que

$$r(t)=\left(r_0-\frac{1}{t+1}\right)\text{ ....}(4)$$

(b) Sustituyendo (4) en (2), se tiene

$$B(t)=B_T\ e^{-\int_t^T\left(r_0-\frac{1}{s+1}\right)ds}$$

$$\therefore\ B(t)=B_T\ e^{-r_0(T-t)}e^{\ln\left(\frac{T+1}{t+1}\right)}$$

$$\therefore\ B(t)=B_T\ \left(\frac{T+1}{t+1}\right)e^{-r_0(T-t)}$$

Se pide $B(T)=B_T=1$.

$$\therefore\ B(t)=e^{-r_0(T-t)}\left(\frac{T+1}{t+1}\right)\text{...}(5)$$

(c) Comparando las ecuaciones (1) y (3), se observa que

$$\delta(t)B(t) = -\frac{T+1}{t+1},$$

con $B(t)$ dada en (5).

$\therefore\ \delta(t) = -e^{r_0(T-t)}$..(6)

(d) Como $\dot{Z} = \delta(t)$, con $\delta(t)$ de (6), por lo tanto

$$Z(t) = \int \delta(t)dt = -\int e^{r_0(T-t)}dt$$

$$\therefore\ Z(t) = \frac{1}{r_0}e^{r_0(T-t)} + C$$

Como $Z(t) = \dfrac{Y(t)}{B(t)}$

$$\therefore\ Z(T) = \frac{Y(T)}{B(T)} = \frac{1}{1} = 1$$

$$\therefore\ Z(T) = 1 = \frac{1}{r_0}e^{0} + C$$

$$\therefore\ C = 1 - \frac{1}{r_0}$$

$$\therefore\ Z(t) = \frac{1}{r_0}e^{r_0(T-t)} + \left(1 - \frac{1}{r_0}\right)$$

$\therefore\ Z(t) = \dfrac{1}{r_0}\left(e^{r_0(T-t)} - 1\right) + 1$..(7)

(e) Queremos resolver la ecuación (3), esto es,

$\dot{Y} - \left(r_0 - \dfrac{1}{t+1}\right)Y = -\dfrac{T+1}{t+1}$....................................(8)

$$\therefore\ \mu(t) = e^{-\int_T^t\left(r_0 - \frac{1}{s+1}\right)ds}$$

$$\therefore\ \mu(t) = e^{-r_0(t-T)}e^{\ln\left(\frac{t+1}{T+1}\right)}$$

$\therefore\ \mu(t) = \dfrac{t+1}{T+1}e^{-r_0(t-T)}$..(9)

Comparando (8) con (5), se tiene

$\mu(t) = \dfrac{1}{B(t)}$..(10)

Multiplicamos (8) por $\mu(t)$ dada en (10):

$$\frac{1}{B(t)}\left[\dot{Y} - \left(r_0 - \frac{1}{t+1}\right)Y\right] = \frac{1}{B(t)}\left[-\frac{T+1}{t+1}\right]$$

$$\therefore\ \frac{d}{dt}\left[\frac{1}{B(t)}Y(t)\right] = -e^{-r_0(t-T)}$$

$$\therefore\ \frac{1}{B(t)}Y(t) = -\int_a^t e^{-r_0(s-T)}\ ds$$

$$\therefore\ \frac{1}{B(T)}Y(T) = -\int_a^T e^{-r_0(s-T)}\ ds$$

$$\therefore\ \frac{1}{B(t)}Y(t) - \frac{1}{B(T)}Y(T) = -\int_a^t e^{-r_0(s-T)}\ ds + \int_a^T e^{-r_0(s-T)}\ ds$$

$$\therefore\ \frac{1}{B(t)}Y(t) - 1 = \int_t^T e^{-r_0(s-T)}\ ds$$

$$\therefore\ Y(t) = B(t)\left[1 + \int_t^T e^{-r_0(s-T)}\ ds\right]$$

$$\therefore\ Y(t) = B(t)\left[1 + \frac{1}{r_0}\left(e^{-r_0(t-T)} - 1\right)\right] \text{..........................(11)}$$

Por último, sustituyendo (5) en (11), y llevando a cabo algunas simplificaciones, se obtiene

$$Y(t) = \frac{T+1}{t+1}\left[\left(1 - \frac{1}{r_0}\right)e^{r_0(t-T)} + \frac{1}{r_0}\right] \text{........................(12)}$$

Observa que, efectivamente, la función $Y(t)$ en (12) satisface $Y(t) = Z(t)B(t)$, con $Z(t)$ y $B(t)$ dadas en (7) y (5), respectivamente.

MATEMÁTICAS APLICADAS A LA ECONOMÍA
TAREA 5 - SOLUCIONES
ECUACIONES DIFERENCIALES I
(SEGUNDA PARTE)
(Temas 4.3-4.4)

1. (a) $y' + 2xy = y$

$$\frac{dy}{dx} = y(1-2x)$$
$$\therefore \int \frac{dy}{y} = \int (1-2x)dx$$
$$\therefore \ln|y| = x - x^2 + C$$
$$\therefore |y| = e^{x-x^2+C} = e^{x-x^2}e^C$$
$$\therefore y = \pm e^{x-x^2}e^C$$
$$\therefore y = Ae^{x-x^2}, \quad A = \pm e^C$$

Solución:

$$y(x) = Ae^{x-x^2}$$

(b) $\dfrac{d\varepsilon}{d\sigma} = \dfrac{\varepsilon}{\sigma(\ln\sigma)}$

$$\int \frac{d\varepsilon}{\varepsilon} = \int \frac{d\sigma}{\sigma(\ln\sigma)}$$
$$\therefore \ln|\varepsilon| = \ln|\ln\sigma| + C$$
$$\therefore |\varepsilon| = e^{\ln|\ln\sigma|+C} = e^{\ln|\ln\sigma|}e^C = |\ln\sigma|\, e^C$$
$$\therefore \varepsilon = \pm(\ln\sigma)\, e^C$$
$$\therefore \varepsilon = A\ln\sigma, \quad A = \pm e^C$$

Solución:

$$\varepsilon(\sigma) = A\ln\sigma$$

(c) $\sqrt{1-t^2}\,\dot{y} = t\sqrt{1-y^2}$

$$\frac{dy}{dt} = \frac{t\sqrt{1-y^2}}{\sqrt{1-t^2}}$$
$$\therefore \int \frac{dy}{\sqrt{1-y^2}} = \int \frac{t}{\sqrt{1-t^2}}dt$$
$$\therefore sen^{-1}y = -\sqrt{1-t^2} + C$$

Solución:

$$y(t) = sen\left(-\sqrt{1-t^2} + C\right)$$

(d) $2p\dfrac{dp}{dx} = \dfrac{1}{x\sqrt{x^2-16}}$

$$\therefore \int 2p\ dp = \int \frac{dx}{x\sqrt{x^2-16}}$$

$\therefore\ p^2 = \dfrac{1}{4}\sec^{-1}\left|\dfrac{x}{4}\right| + C$

Solución:

$$p(x) = \pm\sqrt{\frac{1}{4}\sec^{-1}\left|\frac{x}{4}\right| + C}$$

(e) $(1+x^2)y' - 1 - y^2 = 0$

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\therefore\ \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\therefore\ \tan^{-1} y = \tan^{-1} x + C$$

Solución:

$$y(x) = \tan\left[\tan^{-1} x + C\right]$$

Nota: Aquí no se anulan tan y $\tan^{-1}$, debido al término $+C$

(f) $x^2y' = 1 - x^2 + y^2 - x^2y^2$

$$x^2\frac{dy}{dx} = (1-x^2) + y^2\left(1-x^2\right) = \left(1-x^2\right)\left(1+y^2\right)$$

$$\frac{dy}{dx} = \frac{\left(1-x^2\right)\left(1+y^2\right)}{x^2}$$

$$\therefore\ \int \frac{dy}{1+y^2} = \int \frac{\left(1-x^2\right)}{x^2}dx$$

$$\therefore\ \tan^{-1} y = -\frac{1}{x} - x + C$$

Solución:

$$y(x) = \tan\left(-\frac{1}{x} - x + C\right)$$

2. (a) $y'(\lambda) = ye^{\lambda}, \quad y(0) = -2$

$$\frac{dy}{d\lambda} = ye^{\lambda}$$

$$\therefore \int \frac{dy}{y} = \int e^{\lambda}d\lambda$$

$$\therefore\ \ln|y| = e^{\lambda} + C$$

$$\therefore\ |y| = e^{e^{\lambda}+C} = e^{e^{\lambda}}e^{C}$$

$$\therefore\ y = Ae^{e^{\lambda}}, \quad A = \pm e^{C}$$

Condición inicial:

$$y(0) = -2 = Ae^{e^0} = Ae\ \therefore\ A = -2e^{-1}$$

Solución:

$$y(\lambda) = -2e^{e^{\lambda}-1}$$

(b) $\dot{x} = \dfrac{t^2}{x + t^3 x}, \quad x(0) = -2$

$$\frac{dx}{dt} = \frac{t^2}{x\,(1+t^3)}$$

$$\therefore \int x\ dx = \int \frac{t^2}{1+t^3} dt$$

$$\therefore \frac{x^2}{2} = \frac{1}{3}\ln|1+t^3| + C$$

$$\therefore x = \pm\sqrt{\frac{2}{3}\ln|1+t^3| + 2C}$$

Condición inicial:

$$x(0) = -2 = \pm\sqrt{\frac{2}{3}\ln 1 + 2C} = \pm\sqrt{2C}$$

$\therefore\ 2C = 4$ y tomamos la raíz negativa

Solución:

$$x(t) = -\sqrt{\frac{2}{3}\ln|1+t^3| + 4}$$

(c) $\tan x\ \dfrac{dy}{dx} = y, \quad y\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$$\frac{dy}{dx} = \frac{y}{\tan x}$$

$$\therefore \int \frac{dy}{y} = \int \frac{1}{\tan x}\ dx$$

$$\therefore \int \frac{dy}{y} = \int \frac{\cos x}{sen\, x}\ dx$$

$$\therefore \ln|y| = \ln|sen\, x| + C$$

$$\therefore |y| = e^{\ln|sen\, x| + C} = e^{\ln|sen\, x|}e^C = |sen\, x|\, e^C$$

$$\therefore y = \pm(sen\, x)\, e^C$$

$$\therefore y = A\ sen\, x, \quad A = \pm e^C$$

Condición inicial:

$$y\left(\frac{\pi}{4}\right) = \sqrt{2} = A\ sen\left(\frac{\pi}{4}\right) = A\left(\frac{1}{\sqrt{2}}\right)\ \therefore\ A = 2$$

Solución:

$$y(x) = 2\,sen\, x$$

(d) $2\sqrt{x}\ \dfrac{dy}{dx} = \cos^2 y, \quad y(4) = \dfrac{\pi}{4}$

$$\frac{dy}{dx} = \frac{\cos^2 y}{2\sqrt{x}}$$

$$\therefore \int \frac{dy}{\cos^2 y} = \int \frac{dx}{2\sqrt{x}}$$

$\therefore\ \int \sec^2 y\ dy = \int \dfrac{dx}{2\sqrt{x}}$

$\therefore\ \tan y = \sqrt{x} + C$

$\therefore\ y = \tan^{-1}\left(\sqrt{x} + C\right)$

Condición inicial:

$y(4) = \dfrac{\pi}{4} = \tan^{-1}(2 + C)$

$\therefore\ \tan\left(\dfrac{\pi}{4}\right) = 2 + C$

$\therefore\ 1 = 2 + C\ \ \therefore\ C = -1$

Solución:

$y(x) = \tan^{-1}\left(\sqrt{x} - 1\right)$

(e) $\dot{y} = \dfrac{t^3 + 1}{y^3 + 1}, \quad y(1) = 2$

$\dfrac{dy}{dt} = \dfrac{t^3 + 1}{y^3 + 1}$

$\therefore\ \int (y^3 + 1)\, dy = \int (t^3 + 1)\, dt$

$\therefore\ \dfrac{y^4}{4} + y = \dfrac{t^4}{4} + t + C$

$\therefore\ y^4 + 4y = t^4 + 4t + 4C$

Condición inicial:

$2^4 + 4(2) = 1^4 + 4(1) + 4C\ \ \therefore\ 4C = 19$

Solución (implícita):

$y^4 + 4y = t^4 + 4t + 19$

(f) $\dfrac{dy}{dx} = 2xy^2 + 3x^2y^2, \quad y(1) = -1$

$\dfrac{dy}{dx} = y^2\left(2x + 3x^2\right)$

$\therefore\ \int \dfrac{dy}{y^2} = \int \left(2x + 3x^2\right) dx$

$\therefore\ -\dfrac{1}{y} = x^2 + x^3 + C$

$\therefore\ y = -\left(\dfrac{1}{x^2 + x^3 + C}\right)$

Condición inicial:

$y(1) = -1 = -\left(\dfrac{1}{2 + C}\right)\ \ \therefore\ C = -1$

Solución:

$y(x) = -\left(\dfrac{1}{x^2 + x^3 - 1}\right)$

3. $\dfrac{dC}{dQ} = \dfrac{C}{Q}, \qquad C, Q > 0$

$\therefore\ \int \dfrac{dC}{C} = \int \dfrac{dQ}{Q}$

$\therefore\ \ln C = \ln Q + K$

$\therefore\ C = e^{\ln Q + K} = e^{\ln Q} e^{K}$

$\therefore\ C = AQ, \quad A = e^{K}$

Solución:

$C(Q) = AQ$ (función de costos lineal)

4. $X = AK^{1-\alpha}L^{\alpha},\ \dot{K} = sX,\ L(t) = L_0 e^{\lambda t},$
con $0 < \alpha < 1$,$0 < s < 1$ y $A, L_0, \lambda \in \Re^{+}$. $K(0) = K_0$.

$$\dot{K} = sX = s\left(AK^{1-\alpha}L^{\alpha}\right) = sAK^{1-\alpha}L^{\alpha} = sAK^{1-\alpha}\left(L_0 e^{\lambda t}\right)^{\alpha}$$

$\therefore\ \dfrac{dK}{dt} = sAL_0^{\alpha} e^{\alpha\lambda t} K^{1-\alpha}$

$\therefore \int K^{\alpha-1}\ dK = sAL_0^{\alpha} \int e^{\alpha\lambda t}\ dt$

$\therefore\ \dfrac{K^{\alpha}}{\alpha} = \dfrac{sAL_0^{\alpha}}{\alpha\lambda} e^{\alpha\lambda t} + C$

$\therefore K = \left[\dfrac{sAL_0^{\alpha}}{\lambda} e^{\alpha\lambda t} + \alpha C\right]^{1/\alpha}$

Condición inicial:

$$K(0) = K_0 = \left[\frac{sAL_0^{\alpha}}{\lambda} + \alpha C\right]^{1/\alpha} \quad \therefore \alpha C = K_0^{\alpha} - \frac{sAL_0^{\alpha}}{\lambda}$$

Solución:

$$K(t) = \left[K_0^{\alpha} + \frac{sAL_0^{\alpha}}{\lambda}\left(e^{\alpha\lambda t} - 1\right)\right]^{1/\alpha}$$

5. $\dfrac{dY}{dp} = e^{\alpha p + \beta Y + \gamma},\ \ Y(q) = I,\ \ \alpha, \beta, \gamma, q, I \in \Re^{+}.$

$\dfrac{dY}{dp} = e^{\alpha p + \beta Y + \gamma} = e^{\alpha p + \gamma} e^{\beta Y}$

$\therefore \int\ e^{-\beta Y}\ dY = \int\ e^{\alpha p + \gamma}\ dp$

$\therefore -\dfrac{1}{\beta} e^{-\beta Y} = \dfrac{1}{\alpha} e^{\alpha p + \gamma} + C$

$\therefore e^{-\beta Y} = -\dfrac{\beta}{\alpha} e^{\alpha p + \gamma} - \beta C$

$$\therefore Y = -\frac{1}{\beta}\ln\left(-\frac{\beta}{\alpha}e^{\alpha p+\gamma} - \beta C\right)$$

Condición inicial:

$$Y(q) = I = -\frac{1}{\beta}\ln\left(-\frac{\beta}{\alpha}e^{\alpha q+\gamma} - \beta C\right)$$

$$\therefore e^{-\beta I} = -\frac{\beta}{\alpha}e^{\alpha q+\gamma} - \beta C$$

$$\therefore -\beta C = e^{-\beta I} + \frac{\beta}{\alpha}e^{\alpha q+\gamma}$$

Solución:

$$Y(p) = -\frac{1}{\beta}\ln\left(\frac{\beta}{\alpha}\left(e^{\alpha q+\gamma} - e^{\alpha p+\gamma}\right) + e^{-\beta I}\right)$$

6. $\dfrac{dy}{dx} = \dfrac{y\,(1-\alpha y^{\rho})}{x}, \;\; x > 0,\; 0 < y < \alpha^{-1/\rho},\; \rho \neq 0.$

$$\therefore \int \frac{dy}{y\,(1-\alpha y^{\rho})} = \int \frac{dx}{x}$$

$$\therefore \int \left(\frac{1}{y} + \frac{\alpha y^{\rho-1}}{1-\alpha y^{\rho}}\right) dy = \int \frac{dx}{x}$$

$$\therefore \ln y - \frac{1}{\rho}\ln\left(1-\alpha y^{\rho}\right) = \ln x + C$$

$$\therefore \ln\left[\frac{y}{\left(1-\alpha y^{\rho}\right)^{1/\rho}}\right] = \ln x + C$$

$$\therefore \frac{y}{\left(1-\alpha y^{\rho}\right)^{1/\rho}} = e^{\ln x + C} = e^{\ln x}e^{C}$$

$$\therefore \frac{y}{\left(1-\alpha y^{\rho}\right)^{1/\rho}} = Ax, \quad A = e^{C}$$

$$\therefore \frac{y^{\rho}}{1-\alpha y^{\rho}} = (Ax)^{\rho}$$

$$\therefore y^{\rho} = (Ax)^{\rho}\left[1-\alpha y^{\rho}\right]$$

$$\therefore y^{\rho}\left[1 + (Ax)^{\rho}\alpha\right] = (Ax)^{\rho}$$

$$\therefore y = \left[\frac{(Ax)^{\rho}}{1+(Ax)^{\rho}\alpha}\right]^{1/\rho} = \left[\frac{1}{(Ax)^{-\rho}+\alpha}\right]^{1/\rho}$$

$$\therefore y = \left[\frac{1}{\beta x^{-\rho}+\alpha}\right]^{1/\rho}, \quad \beta = A^{-\rho}$$

Solución:

$$y(x) = \left[\beta x^{-\rho} + \alpha\right]^{-1/\rho}$$

7. $\dfrac{dx}{dt} + a(t)x = 0$, con $a(t) = a + bc^t$, donde $a, b, c > 0$, $c \neq 1$.

$\therefore \dfrac{dx}{dt} = -\left(a + bc^t\right) x$

$\therefore \int \dfrac{dx}{x} = -\int \left(a + bc^t\right) dt$

$\therefore \ln x = -\left(at + \dfrac{bc^t}{\ln c}\right) + K$

$\therefore x = e^{-\left(at + bc^t/\ln c\right) + K} = e^{-at}\, e^{-bc^t/\ln c}\, e^K = \left(e^{-a}\right)^t \left(e^{-b/\ln c}\right)^{c^t} \left(e^K\right)$

Definimos $p = e^{-a}, q = e^{-b/\ln c}$ y $C = e^K$.

$\therefore x(t) = Cp^t q^{c^t}$.

8. (a) $\dfrac{dr}{d\theta} = r - 4\theta + 4$

Sea $u = r - 4\theta$ (esto es, $u(\theta) = r(\theta) - 4\theta$)

$\therefore \dfrac{du}{d\theta} = \dfrac{d}{d\theta}(r - 4\theta) = \dfrac{dr}{d\theta} - 4 = (r - 4\theta + 4) - 4 = r - 4\theta = u$

$\therefore \dfrac{du}{d\theta} = u$

$\therefore u = Ae^\theta$

$\therefore r - 4\theta = Ae^\theta$

Solución:

$r(\theta) = Ae^\theta + 4\theta$

(b) $t\dfrac{dx}{dt} = e^{-xt} - x, \quad x(1) = 0$

Sea $u = xt$ (esto es, $u(t) = x(t)t$)

$\therefore \dfrac{du}{dt} = \dfrac{d}{dt}(xt) = x + t\dfrac{dx}{dt} = x + \left(e^{-xt} - x\right) = e^{-xt} = e^{-u}$

$\therefore \dfrac{du}{dt} = e^{-u}$

$\therefore \int e^u du = \int dt$

$\therefore e^u = t + C$

$\therefore u = \ln(t + C)$

$\therefore xt = \ln(t + C)$

$\therefore x = \dfrac{\ln(t + C)}{t}$

Condiciòn inicial:

$\therefore x(1) = 0 = \ln(1 + C) \quad \therefore \ C = 0$

Solución:

$x(t) = \dfrac{\ln t}{t}$

(c) $(x+e^y)\,y' = xe^{-y} - 1$

Sea $u = x + e^y$ (esto es, $u(x) = x + e^{y(x)}$)

$\therefore \dfrac{du}{dx} = \dfrac{d}{dx}(x+e^y) = 1 + e^y \dfrac{dy}{dx} = 1 + e^y \left[\dfrac{xe^{-y}-1}{x+e^y}\right]$

$\therefore \dfrac{du}{dx} = \dfrac{x + e^y + e^y(xe^{-y}-1)}{x+e^y} = \dfrac{2x}{x+e^y} = \dfrac{2x}{u}$

$\therefore \dfrac{du}{dx} = \dfrac{2x}{u}$

$\therefore \int u\ du = \int 2x\ dx$

$\therefore \dfrac{u^2}{2} = x^2 + C$

$\therefore u = \pm\sqrt{2x^2 + 2C}$

$\therefore x + e^y = \pm\sqrt{2x^2 + 2C}$

$\therefore e^y = \pm\sqrt{2x^2 + 2C} - x$

Solución:

$y(x) = \ln\left[\pm\sqrt{2x^2+2C} - x\right]$

(d) $e^x \dfrac{dx}{dt} = 2e^t\sqrt{16 - e^{2x}}$

Sea $u = e^x$ (esto es, $u(t) = e^{x(t)}$)

$\therefore \dfrac{du}{dt} = \dfrac{de^x}{dt} = e^x \dfrac{dx}{dt} = 2e^t\sqrt{16 - e^{2x}} = 2e^t\sqrt{16-u^2}$

$\therefore \dfrac{du}{dt} = 2e^t\sqrt{16-u^2}$

$\therefore \int \dfrac{du}{\sqrt{16-u^2}} = \int 2e^t dt$

$\therefore sen^{-1}\left(\dfrac{u}{4}\right) = 2e^t + C$

$\therefore u = 4\,sen\,(2e^t + C)$

$\therefore e^x = 4\,sen\,(2e^t + C)$

Solución:

$x(t) = \ln\left[4sen\,(2e^t + C)\right]$

(e) $\dfrac{dx}{dt} = -\dfrac{x}{t} + \dfrac{xt}{\ln(xt)}, \quad x, t > 1$

Sea $u = \ln(xt)$ (esto es, $u(t) = \ln(x(t)t)$)

$\therefore \dfrac{du}{dt} = \dfrac{d\ln(xt)}{dt} = \dfrac{d(\ln x + \ln t)}{dt} = \dfrac{1}{x}\dfrac{dx}{dt} + \dfrac{1}{t}$

$\therefore \dfrac{du}{dt} = \dfrac{1}{x}\left[-\dfrac{x}{t} + \dfrac{xt}{\ln(xt)}\right] + \dfrac{1}{t} = \dfrac{t}{\ln(xt)} = \dfrac{t}{u}$

$\therefore \dfrac{du}{dt} = \dfrac{t}{u}$

$\therefore \int \ u \ du = \int \ t \ dt$

$\therefore \dfrac{u^2}{2} = \dfrac{t^2}{2} + C$

$\therefore u = \pm\sqrt{t^2 + 2C}$

$\therefore \ln(xt) = \pm\sqrt{t^2 + 2C}$

$\therefore xt = e^{\pm\sqrt{t^2+2C}}$

Solución:

$$x(t) = \frac{e^{\pm\sqrt{t^2+2C}}}{t}$$

(f) $\dfrac{dx}{dt} = \dfrac{2x}{t} + t\sec\left(\dfrac{x}{t^2}\right)$

Sea $u = \dfrac{x}{t^2}$ (esto es, $u(t) = \dfrac{x(t)}{t^2}$)

$\therefore \dfrac{du}{dt} = \dfrac{d}{dt}\left(\dfrac{x}{t^2}\right) = -\dfrac{2x}{t^3} + \dfrac{1}{t^2}\left(\dfrac{dx}{dt}\right) = -\dfrac{2x}{t^3} + \dfrac{1}{t^2}\left(\dfrac{2x}{t} + t\sec u\right)$

$\therefore \dfrac{du}{dt} = \dfrac{1}{t}\sec u$

$\therefore \int \dfrac{du}{\sec u} = \int \dfrac{dt}{t}$

$\therefore \int \cos u \ du = \int \dfrac{dt}{t}$

$\therefore sen\, u = \ln|t| + C$

$\therefore u = sen^{-1}(\ln|t| + C)$

$\therefore \dfrac{x}{t^2} = sen^{-1}(\ln|t| + C)$

Solución:

$$x(t) = t^2 sen^{-1}(\ln|t| + C)$$

(g) $\dfrac{dx}{dt} = \dfrac{2t}{x + t^2}$

Sea $u = x + t^2$ (esto es, $u(t) = x(t) + t^2$)

$\therefore \dfrac{du}{dt} = \dfrac{d(x + t^2)}{dt} = \dfrac{dx}{dt} + 2t = \dfrac{2t}{x + t^2} + 2t = \dfrac{2t}{u} + 2t$

$\therefore \dfrac{du}{dt} = 2t\left(\dfrac{1}{u} + 1\right) = 2t\left(\dfrac{1+u}{u}\right)$

$\therefore \int \dfrac{u}{1+u} du = \int \ 2t \ dt$

$\therefore \int \left(1 - \dfrac{1}{1+u}\right) du = \int \ 2t \ dt$

$\therefore u - \ln|1+u| = t^2 + C$

$\therefore x + t^2 - \ln|1 + x + t^2| = t^2 + C$

Solución (implícita):

$x - \ln|1 + x + t^2| = C$

9. (a) Ecuación separable:

$\frac{dv}{dt} = 9e^v - 1$

$\therefore \int \frac{1}{9e^v - 1} dv = \int dt$

$\therefore \int \frac{1}{9e^v - 1} \left(\frac{e^{-v}}{e^{-v}}\right) dv = \int dt$

$\therefore \int \frac{e^{-v}}{9 - e^{-v}} dv = \int dt$

$\therefore \ln|9 - e^{-v}| = t + C$

$\therefore |9 - e^{-v}| = e^{t+C} = e^t e^C$

$\therefore 9 - e^{-v} = \pm e^t e^C = Ae^t, \quad A = \pm e^C$

$\therefore e^{-v} = 9 - Ae^t$

$\therefore -v = \ln(9 - Ae^t)$

Solución:

$v(t) = -\ln(9 - Ae^t)$

(b) Sustitución:

Sea $u = e^{-v}$ (esto es, $u(t) = e^{-v(t)}$)

$\therefore \frac{du}{dt} = \frac{de^{-v}}{dt} = -e^{-v}\frac{dv}{dt} = -e^{-v}(9e^v - 1) = -9 + e^{-v}$

$\therefore \frac{du}{dt} = -9 + u \longleftarrow$ Ec. lineal autónoma

$\therefore u = Ce^t + 9$

$\therefore e^{-v} = Ce^t + 9$

Solución:

$v(t) = -\ln(Ce^t + 9)$

Este resultado equivale al del inciso anterior, definiendo $C = -A$.

10. (a) $\frac{dy}{dx} + P(x)y = Q(x)(y \ln y)$

Sea $u = \ln y$ (esto es, $u(x) = \ln y(x)$)

$\therefore \frac{du}{dx} = \frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{y}[Q(x)(y \ln y) - P(x)y]$

$\therefore \frac{du}{dx} = Q(x)\ln y - P(x)$

$\therefore \frac{du}{dx} = Q(x)u - P(x)$

$\therefore \frac{du}{dx} - Q(x)u = -P(x) \longleftarrow$ Ec. lineal

(b) Escribimos la ecuación en la forma

$$\frac{dy}{dx} + (-4x)\,y = \left(-\frac{2}{x}\right)(y \ln y)\,,$$

de donde

$$P\,(x) = -4x,\;\; Q\,(x) = -\frac{2}{x}.$$

De esta manera, se obtiene la ecuación lineal

$$\frac{du}{dx} + \frac{2}{x}u = 4x$$

$$\mu\,(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$$

$$\therefore \frac{d}{dx}\left[x^2 u\right] = 4x^3$$

$$\therefore x^2 u = \int 4x^3 dx = x^4 + C$$

$$\therefore u = x^2 + \frac{C}{x^2}$$

$$\therefore \ln y = x^2 + \frac{C}{x^2}$$

Solución:

$$y = e^{x^2 + C/x^2}$$

11. $t\dot{x} = x - f(t)x^2, \qquad t > 0$

(a) Sea $u = \dfrac{x}{t}$ (esto es, $u(t) = \dfrac{x(t)}{t}$)

$$\therefore \frac{du}{dt} = \frac{d}{dt}\left(\frac{x}{t}\right) = -\frac{x}{t^2} + \frac{1}{t}\left(\frac{dx}{dt}\right) = -\frac{x}{t^2} + \frac{1}{t}\left(\frac{x - f(t)x^2}{t}\right)$$

$$\therefore \frac{du}{dt} = -\frac{f(t)x^2}{t^2} = -f(t)\left(\frac{x}{t}\right)^2 = -f(t)u^2$$

$$\therefore \frac{du}{dt} = -f(t)u^2 \longleftarrow \text{Ec. separable}$$

(b) Sea $f(t) = \dfrac{t^3}{t^4+2}$

$$\therefore \frac{du}{dt} = -f(t)u^2 = -\left(\frac{t^3}{t^4+2}\right)u^2$$

$$\therefore \int u^{-2}du = -\int \frac{t^3\;dt}{t^4+2}$$

$$\therefore -\frac{1}{u} = -\frac{1}{4}\ln|t^4+2| - C = -\frac{1}{4}\ln(t^4+2) - C = -\left(\frac{4\ln(t^4+2) + 4C}{4}\right)$$

$$\therefore u = \frac{4}{\ln(t^4+2) + 4C}$$

$$\therefore \frac{x}{t} = \frac{4}{\ln(t^4+2) + 4C}$$

$\therefore x = \dfrac{4t}{\ln(t^4+2)+4C}$

Condición inicial:

$x(1) = 1 = \dfrac{4}{\ln 3 + 4C} \quad \therefore 4C = 4 - \ln 3$

Solución:

$$\therefore x(t) = \frac{4t}{\ln(t^4+2)+4-\ln(3)} = \frac{4t}{\ln\left(\dfrac{t^4+2}{3}\right)+4}$$

12. $\dot{x} = g(x/t), \qquad t > 0$

(a) Sea $u = \dfrac{x}{t}$ (esto es, $u(t) = \dfrac{x(t)}{t}$)

$$\therefore \frac{du}{dt} = \frac{d}{dt}\left(\frac{x}{t}\right) = -\frac{x}{t^2} + \frac{1}{t}\left(\frac{dx}{dt}\right) = -\frac{x}{t^2} + \frac{1}{t}g\left(\frac{x}{t}\right)$$

$$\therefore \frac{du}{dt} = \frac{1}{t}\left[-\frac{x}{t} + g\left(\frac{x}{t}\right)\right] = \frac{1}{t}\left[-u + g(u)\right]$$

$$\therefore \frac{du}{dt} = \frac{g(u)-u}{t} \longleftarrow \text{Ec. separable}$$

(b) $\dot{x} = 1 + (x/t) - (x/t)^2$.

$g(u) = 1 + u - u^2$

$$\therefore \frac{du}{dt} = \frac{g(u)-u}{t} = \frac{1-u^2}{t}$$

$$\therefore \int \frac{du}{1-u^2} = \int \frac{dt}{t}$$

$$\therefore \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| = \ln|t| + C$$

$$\therefore \ln\left|\frac{u+1}{u-1}\right| = 2\ln|t| + 2C$$

$$\therefore \left|\frac{u+1}{u-1}\right| = e^{2\ln|t|+2C} = e^{\ln|t|^2}e^{2C} = t^2e^{2C}$$

$$\therefore \frac{u+1}{u-1} = \pm e^{2C}t^2 = At^2, \quad A = \pm e^{2C}$$

$$\therefore u + 1 = At^2(u-1)$$

$$\therefore u(1-At^2) = -(At^2+1)$$

$$\therefore u = -\frac{At^2+1}{1-At^2} = \frac{At^2+1}{At^2-1}$$

$$\therefore \frac{x}{t} = \frac{At^2+1}{At^2-1}$$

Solución:

$$x(t) = \frac{t(At^2+1)}{At^2-1}$$

13. (a) $\dot{x} + 2x = -x^2$

$\dot{x} = -2x - x^2 \longleftarrow$Ec. de Bernoulli $(n = 2)$

Sea $u = x^{1-2} = x^{-1}$ (esto es, $u(t) = [x(t)]^{-1}$)

$$\therefore \frac{du}{dt} = -x^{-2}\frac{dx}{dt} = -x^{-2}\left[-2x - x^2\right] = 2x^{-1} + 1 = 2u + 1$$

$$\therefore \frac{du}{dt} - 2u = 1 \longleftarrow \text{Ec. lineal autónoma}$$

$$\therefore u = Ae^{2t} - \frac{1}{2}$$

$$\therefore \frac{1}{x} = Ae^{2t} - \frac{1}{2} = \frac{2Ae^{2t} - 1}{2}$$

Solución:

$$x(t) = \frac{2}{2Ae^{2t} - 1}$$

(b) $\dot{x} - \frac{1}{t}x = -\frac{1}{t}x^2$

$\dot{x} = \frac{1}{t}x - \frac{1}{t}x^2 \longleftarrow$Ec. de Bernoulli $(n = 2)$

Sea $u = x^{1-2} = x^{-1}$ (esto es, $u(t) = [x(t)]^{-1}$)

$$\therefore \frac{du}{dt} = -x^{-2}\frac{dx}{dt} = -x^{-2}\left[\frac{1}{t}x - \frac{1}{t}x^2\right] = -\frac{x^{-1}}{t} + \frac{1}{t} = \frac{1}{t}(-u + 1)$$

$$\therefore \frac{du}{dt} = \frac{1}{t}(1 - u) \longleftarrow \text{Ec. lineal no autónoma / Ec. separable}$$

$$\therefore u = \frac{A}{t} + 1$$

$$\therefore \frac{1}{x} = \frac{A}{t} + 1$$

Solución:

$$x(t) = \frac{1}{1 + At^{-1}} = \frac{t}{t + A}$$

(c) $x\frac{dx}{dt} = x^2 + x^4, \quad x(0) = -1$

$\dot{x} = x + x^3 \longleftarrow$Ec. de Bernoulli $(n = 3)$

Sea $u = x^{1-3} = x^{-2}$ (esto es, $u(t) = [x(t)]^{-2}$)

$$\therefore \frac{du}{dt} = -2x^{-3}\frac{dx}{dt} = -2x^{-3}\left[x + x^3\right] = -2x^{-2} - 2 = -2u - 2$$

$$\therefore \frac{du}{dt} = -2u - 2 \longleftarrow \text{Ec. lineal autónoma}$$

$$\therefore u = Ae^{-2t} - 1$$

$$\therefore \frac{1}{x^2} = Ae^{-2t} - 1$$

$$\therefore x = \pm\sqrt{\frac{1}{Ae^{-2t} - 1}}$$

Condición inicial:

$$x(0) = -1 = \pm\sqrt{\frac{1}{A-1}}$$

$\therefore\ A = 2$ y tomamos la raíz negativa

Solución:

$$x(t) = -\sqrt{\frac{1}{2e^{-2t}-1}}$$

(d) $\dot{x} = \dfrac{2x}{t} - 5x^2t^2, \quad x(1) = \dfrac{1}{2}$

$$\dot{x} = \frac{2x}{t} - 5t^2x^2 \longleftarrow \text{Ec. de Bernoulli } (n=2)$$

Sea $u = x^{1-2} = x^{-1}$ (esto es, $u(t) = [x(t)]^{-1}$)

$$\therefore \frac{du}{dt} = -x^{-2}\frac{dx}{dt} = -x^{-2}\left[\frac{2x}{t} - 5t^2x^2\right] = \left[-\frac{2x^{-1}}{t} + 5t^2\right]$$

$$\therefore \frac{du}{dt} = -\frac{2u}{t} + 5t^2 \longleftarrow \text{Ec. lineal no autónoma}$$

$$\therefore u = t^3 + Ct^{-2} = \frac{t^5+C}{t^2}$$

$$\therefore \frac{1}{x} = \frac{t^5+C}{t^2}$$

$$\therefore\ x = \frac{t^2}{t^5+C}$$

Condición inicial

$$x(1) = \frac{1}{2} = \frac{1}{1+C} \quad \therefore\ C = 1$$

Solución:

$$x(t) = \frac{t^2}{t^5+1}$$

(e) $2y\dot{y} - y^2 = e^{3t}, \quad y(0) = -1$

$$\dot{y} = \frac{1}{2}y + \frac{e^{3t}}{2}y^{-1} \longleftarrow \text{Ec. de Bernoulli } (n=-1)$$

Sea $u = y^{1-(-1)} = y^2$ (esto es, $u(t) = [y(t)]^2$)

$$\therefore \frac{du}{dt} = 2y\frac{dy}{dt} = 2y\left[\frac{1}{2}y + \frac{e^{3t}}{2}y^{-1}\right] = y^2 + e^{3t} = u + e^{3t}$$

$$\therefore \frac{du}{dt} - u = e^{3t} \longleftarrow \text{Ec. lineal no autónoma}$$

$$\therefore u = \frac{1}{2}e^{3t} + Ae^t$$

$$\therefore y^2 = \frac{1}{2}e^{3t} + Ae^t$$

$$\therefore y = \pm\sqrt{\frac{1}{2}e^{3t} + Ae^t}$$

Condición inicial:

$$y(0) = -1 = \pm\sqrt{\frac{1}{2} + A}$$

$\therefore\ A = \frac{1}{2}$ y tomamos la raíz negativa

Solución:

$$y(t) = -\sqrt{\frac{1}{2}e^{3t} + \frac{1}{2}e^{t}}$$

(f) $\dfrac{dy}{dx} + y = y^7, \quad y(0) = 1$

$\dfrac{dy}{dx} = -y + y^7 \longleftarrow$ Ecuación de Bernoulli $(n = 7)$

Sea $u = y^{1-7} = y^{-6}$ (esto es, $u(x) = [y(x)]^{-6}$)

$$\therefore \frac{du}{dx} = -6y^{-7}\frac{dy}{dx} = -6y^{-7}\left[-y + y^7\right] = 6y^{-6} - 6 = 6u - 6$$

$\therefore \dfrac{du}{dx} - 6u = -6 \longleftarrow$ Ec. lineal autónoma

$\therefore u = Ae^{6x} + 1$

$\therefore \dfrac{1}{y^6} = Ae^{6x} + 1$

$$\therefore y = \pm\frac{1}{(Ae^{6x} + 1)^{1/6}}$$

Condiciòn inicial:

$$y(0) = 1 = \pm\frac{1}{(A + 1)^{1/6}}$$

$\therefore A = 0$ y tomamos la raíz positiva

Solución:

$$y(x) = 1$$

14. $\dot{x} = tx + \dfrac{t}{x}$

(a) Ecuación de Bernoulli:

$\dot{x} = tx + tx^{-1} \longleftarrow$ Ec. de Bernoulli $(n = -1)$

Sea $u = x^{1-(-1)} = x^2$ (esto es, $u(t) = [x(t)]^2$)

$$\therefore \frac{du}{dt} = 2x\frac{dx}{dt} = 2x\left[tx + tx^{-1}\right] = 2tx^2 + 2t = 2t\left(x^2 + 1\right)$$

$\therefore \dfrac{du}{dt} = 2t(u+1) \longleftarrow$ Ec. lineal no autónoma / Ec. separable

$\therefore u = Ce^{t^2} - 1$

$\therefore x^2 = Ce^{t^2} - 1$

Solución:

$$x(t) = \pm\sqrt{Ce^{t^2} - 1}$$

(b) Ecuación separable:

$$\frac{dx}{dt} = t\left(x + \frac{1}{x}\right) = t\left(\frac{x^2+1}{x}\right)$$

$$\therefore \int \frac{x}{x^2+1}dx = \int t\ dt$$

$$\therefore \frac{1}{2}\ln(x^2+1) = \frac{t^2}{2} + C$$

$$\therefore x^2 + 1 = e^{t^2+2C} = e^{t^2}e^{2C} = Ae^{t^2}, \quad A = e^{2C}$$

Solución:

$$x(t) = \pm\sqrt{Ae^{t^2} - 1}$$

15. $\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{\alpha y^{\rho+1}}{x} \longleftarrow$ Ec. de Bernoulli $(n = \rho + 1)$

Sea $u = y^{1-(\rho+1)} = y^{-\rho}$ (esto es, $u(x) = [y(x)]^{-\rho}$)

$$\therefore \frac{du}{dx} = -\rho y^{-\rho-1}\frac{dy}{dx} = -\rho y^{-\rho-1}\left[\frac{y}{x} - \frac{\alpha y^{\rho+1}}{x}\right] = -\frac{\rho}{x}y^{-\rho} + \frac{\rho\alpha}{x}$$

$$\therefore \frac{du}{dx} + \frac{\rho}{x}u = \frac{\rho\alpha}{x} \longleftarrow \text{Ec. lineal no autónoma / Ec. separable}$$

$$\therefore u = Cx^{-\rho} + \alpha$$

$$\therefore y^{-\rho} = Cx^{-\rho} + \alpha$$

Solución:

$$y(x) = (Cx^{-\rho} + \alpha)^{-1/\rho}$$

16. (a) $\dot{x} = (x-t)^2 - (x-t) + 1$

Sea $u = x - t$ (esto es, $u(t) = x(t) - t$)

$$\therefore \frac{du}{dt} = \frac{d}{dt}(x-t) = \dot{x} - 1 = \left[(x-t)^2 - (x-t) + 1\right] - 1$$

$$\therefore \frac{du}{dt} = (x-t)^2 - (x-t) = u^2 - u$$

$$\therefore \frac{du}{dt} = -u + u^2 \longleftarrow \text{Ec. de Bernoulli } (n = 2)$$

Sea $w = u^{1-2} = u^{-1}$ (esto es, $w(t) = [u(t)]^{-1}$)

$$\therefore \frac{dw}{dt} = -u^{-2}\frac{du}{dt} = -u^{-2}\left[-u + u^2\right] = u^{-1} - 1 = w - 1$$

$$\therefore \frac{dw}{dt} - w = -1 \longleftarrow \text{Ec. lineal autónoma}$$

$$\therefore w = Ae^t + 1$$

$$\therefore u = \frac{1}{Ae^t + 1}$$

$$\therefore x - t = \frac{1}{Ae^t + 1}$$

Solución:

$$x(t) = \frac{1}{Ae^t + 1} + t$$

(b) $\dot{x} = 2x + \frac{x}{t} + \frac{x^2}{t}$

Sea $u = \frac{x}{t}$ (esto es, $u(t) = \frac{x(t)}{t}$)

$\therefore \frac{du}{dt} = \frac{d}{dt}\left(\frac{x}{t}\right) = -\frac{x}{t^2} + \frac{1}{t}\dot{x} = -\frac{x}{t^2} + \frac{1}{t}\left[2x + \frac{x}{t} + \frac{x^2}{t}\right]$

$\therefore \frac{du}{dt} = 2\frac{x}{t} + \frac{x^2}{t^2} = 2u + u^2 \longleftarrow$Ec. de Bernoulli $(n = 2)$

Sea $w = u^{1-2} = u^{-1}$ (esto es, $w(t) = [u(t)]^{-1}$)

$\therefore \frac{dw}{dt} = -u^{-2}\frac{du}{dt} = -u^{-2}\left[2u + u^2\right] = -2u^{-1} - 1 = -2w - 1$

$\therefore \frac{dw}{dt} + 2w = -1 \longleftarrow$Ec. lineal autónoma

$\therefore w = Ae^{-2t} - \frac{1}{2}$

$\therefore \frac{1}{u} = Ae^{-2t} - \frac{1}{2} = \frac{2Ae^{-2t} - 1}{2}$

$\therefore \frac{x}{t} = \frac{2}{2Ae^{-2t} - 1}$

Solución:

$$x(t) = \frac{2t}{2Ae^{-2t} - 1}$$

(c) $\dot{x} = 1 + e^x$

Sea $u = e^x$ (esto es, $u(t) = e^{x(t)}$)

$\therefore \frac{du}{dt} = \frac{d}{dt}\left(e^x\right) = e^x\dot{x} = e^x\left[1 + e^x\right] = e^x + e^{2x} = u + u^2$

$\therefore \frac{du}{dt} = u + u^2 \longleftarrow$Ec. de Bernoulli $(n = 2)$

Sea $w = u^{1-2} = u^{-1}$ (esto es, $w(t) = [u(t)]^{-1}$)

$\therefore \frac{dw}{dt} = -u^{-2}\frac{du}{dt} = -u^{-2}\left[u + u^2\right] = -u^{-1} - 1 = -w - 1$

$\therefore \frac{dw}{dt} + w = -1 \longleftarrow$Ec. lineal autónoma

$\therefore w = Ae^{-t} - 1$

$\therefore \frac{1}{u} = Ae^{-t} - 1$

$\therefore e^x = \frac{1}{Ae^{-t} - 1}$

Solución:

$$x(t) = \ln\left(\frac{1}{Ae^{-t} - 1}\right) = -\ln\left(Ae^{-t} - 1\right)$$

17. $\dot{N} = kN\left(N^* - N\right), \quad N\left(0\right) = N_0, \quad 0 \leq N_0 \leq N^*,\ k > 0$

$\dot{N} = kN\left(N^* - N\right) = \left(kN^*\right)N - kN^2 \longleftarrow$ Ec. de Bernoulli $(n = 2)$

Sea $u = N^{1-2} = N^{-1}$ (esto es, $u(t) = [N(t)]^{-1}$)

$$\therefore \frac{du}{dt} = -N^{-2}\dot{N} = -N^{-2}\left[\left(kN^*\right)N - kN^2\right]$$

$$\therefore \frac{du}{dt} = -\left(kN^*\right)N^{-1} + k = -\left(kN^*\right)u + k$$

$$\therefore \frac{du}{dt} + \left(kN^*\right)u = k \longleftarrow \text{Ec. lineal autónoma}$$

$$\therefore u = Ae^{-kN^*t} + \frac{1}{N^*}$$

$$\therefore \frac{1}{N} = Ae^{-kN^*t} + \frac{1}{N^*}$$

$$\therefore N = \frac{1}{Ae^{-kN^*t} + \frac{1}{N^*}}$$

Condición inicial:

$$N\left(0\right) = N_0 = \frac{1}{A + \frac{1}{N^*}} \quad \therefore A = \frac{1}{N_0} - \frac{1}{N^*}$$

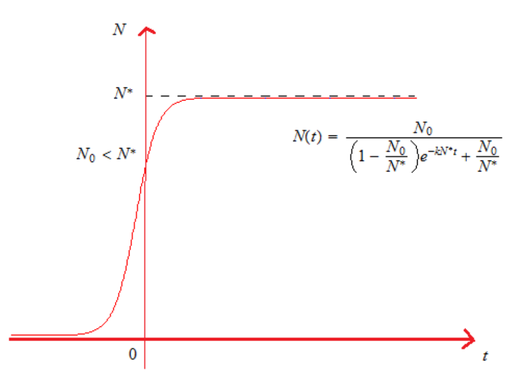
Solución:

$$N(t) = \frac{1}{\left(\frac{1}{N_0} - \frac{1}{N^*}\right)e^{-kN^*t} + \frac{1}{N^*}} = \frac{N_0}{\left(1 - \frac{N_0}{N^*}\right)e^{-kN^*t} + \frac{N_0}{N^*}}$$

Largo plazo:

$$\lim_{t\to\infty} N\left(t\right) = N^*$$

Así, a la larga todo el pueblo se habrá enterado del chisme.



18. $\dot{k} = sf(k) - (n+\delta)k,\ 0 < \alpha < 1, 0 < s < 1,\ f(k) = k^{\alpha}$

$\dot{k} = sf(k) - (n+\delta)k = sk^{\alpha} - (n+\delta)k \longleftarrow$Ec. de Bernoulli

Sea $u = k^{1-\alpha}$ (esto es, $u(t) = [k(t)]^{1-\alpha}$)

$$\therefore \frac{du}{dt} = (1-\alpha)k^{-\alpha}\dot{k} = (1-\alpha)k^{-\alpha}\left[sk^{\alpha} - (n+\delta)k\right]$$

$$\therefore \frac{du}{dt} = (1-\alpha)s - (1-\alpha)(n+\delta)k^{1-\alpha}$$

$$\therefore \frac{du}{dt} = (1-\alpha)s - (1-\alpha)(n+\delta)u$$

$$\therefore \frac{du}{dt} + (1-\alpha)(n+\delta)u = (1-\alpha)s \longleftarrow \text{Ec. lineal autónoma}$$

$$\therefore u = Ae^{-(1-\alpha)(n+\delta)t} + \frac{s}{n+\delta}$$

$$\therefore k^{1-\alpha} = Ae^{-(1-\alpha)(n+\delta)t} + \frac{s}{n+\delta}$$

Solución:

$$k(t) = \left[Ce^{-(1-\alpha)(n+\delta)t} + \frac{s}{n+\delta}\right]^{\frac{1}{1-\alpha}}$$

Largo plazo:

$$\lim_{t\to\infty} k(t) = \lim_{t\to\infty}\left[Ce^{-(1-\alpha)(n+\delta)t} + \frac{s}{n+\delta}\right]^{\frac{1}{1-\alpha}} = \left[\frac{s}{n+\delta}\right]^{\frac{1}{1-\alpha}}$$

Así, el capital se estabiliza en el valor $\left[\frac{s}{n+\delta}\right]^{\frac{1}{1-\alpha}}$.

19. $\dot{k} = \alpha A(n_0)^{\alpha}e^{(\alpha v+\epsilon)t}k^{b} - \alpha\delta k,$

$\alpha v + \epsilon + \alpha\delta(1-b) \neq 0,\ \ A, n_0, a, b, v, \alpha, \delta, \epsilon \in \Re^{+}$

$\dot{k} = \alpha A(n_0)^{\alpha}e^{(\alpha v+\epsilon)t}k^{b} - \alpha\delta k \longleftarrow$Ec. de Bernoulli

Sea $u = k^{1-b}$ (esto es, $u(t) = [k(t)]^{1-b}$)

$$\therefore \frac{du}{dt} = (1-b)k^{-b}\dot{k} = (1-b)k^{-b}\left[\alpha A(n_0)^{\alpha}e^{(\alpha v+\epsilon)t}k^{b} - \alpha\delta k\right]$$

$$\therefore \frac{du}{dt} = (1-b)\alpha A(n_0)^{\alpha}e^{(\alpha v+\epsilon)t} - \alpha\delta(1-b)k^{1-b}$$

$$\therefore \frac{du}{dt} = (1-b)\alpha A(n_0)^{\alpha}e^{(\alpha v+\epsilon)t} - \alpha\delta(1-b)u$$

$$\therefore \frac{du}{dt} + \alpha\delta(1-b)u = (1-b)\alpha A(n_0)^{\alpha}e^{(\alpha v+\epsilon)t} \longleftarrow \text{Lineal no autónoma}$$

$$\therefore u = \frac{(1-b)\alpha A(n_0)^{\alpha}}{\alpha v + \epsilon + \alpha\delta(1-b)}e^{(\alpha v+\epsilon)t} + Ce^{-\alpha\delta(1-b)t}$$

$$\therefore k^{1-b} = \frac{(1-b)\,\alpha A(n_0)^{\alpha}}{\alpha v + \epsilon + \alpha\delta\,(1-b)} e^{(\alpha v+\epsilon)t} + Ce^{-\alpha\delta(1-b)t}$$

Solución:

$$k(t) = \left(\frac{(1-b)\,\alpha A(n_0)^{\alpha}}{\alpha v + \epsilon + \alpha\delta\,(1-b)} e^{(\alpha v+\epsilon)t} + Ce^{-\alpha\delta(1-b)t}\right)^{\frac{1}{1-b}}$$

20. $\dot{P} = P\,(b - a\ln P)$, $P(t) > 0$, $a, b > 0$

(a) Puntos fijos $\left(\dot{P} = 0\right)$:

$P^*\,(b - a\ln P^*) = 0$

$\therefore b - a\ln P^* = 0$ ($P^* = 0$ no está en el dominio)

$\therefore \dfrac{b}{a} = \ln P^*$

$\therefore P^* = e^{b/a}$

(b) Sea $x = \ln P$ (esto es, $x(t) = \ln P(t)$)

$\therefore \dot{x} = \dfrac{1}{P}\dot{P} = \dfrac{1}{P}\left[P\,(b - a\ln P)\right] = b - a\ln P = b - ax$

$\therefore \dot{x} + ax = b \longleftarrow$Ecuación lineal autónoma

Solución:

$x(t) = Ae^{-at} + \dfrac{b}{a}$

(c) $x = \ln P \Longrightarrow P = e^{x}$

$\therefore P\,(t) = e^{Ae^{-at}+\frac{b}{a}}$

Condición inicial:

$P(0) = 1 = e^{A+\frac{b}{a}} \quad \therefore A = -\dfrac{b}{a}$

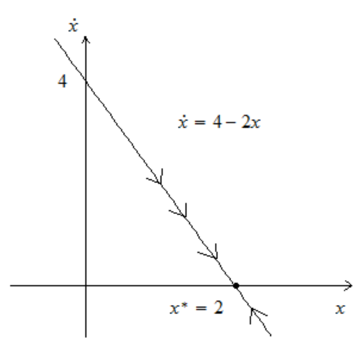
Solución:

$P\,(t) = e^{\frac{b}{a}-\frac{b}{a}e^{-at}}$

(d) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} e^{\frac{b}{a}-\frac{b}{a}e^{-at}} = e^{\frac{b}{a}} = P^*$

21. (a) $\dot{x} = 4 - 2x = f\,(x)$

Diagrama de fase:



Puntos fijos ($\dot{x} = 0$):

$x^* = 2$

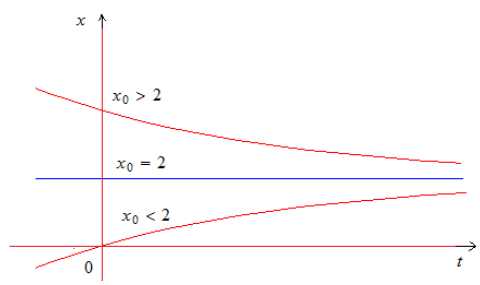
Clasificación:

$f'(x) = -2$

$\therefore f'(2) = -2 < 0$

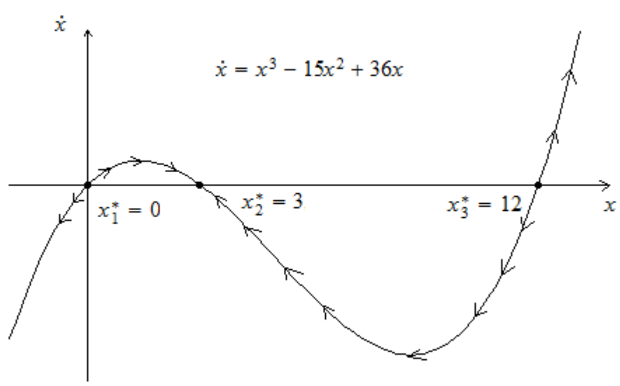
$\therefore x^* = 2$ es un punto de equilibrio estable.

Gráfica de las soluciones:



(b) $\dot{x} = x^3 - 15x^2 + 36x = x(x-3)(x-12) = f(x)$

Diagrama de fase:



Puntos fijos ($\dot{x} = 0$):

$x_1^* = 0,\ x_2^* = 3,\ x_3^* = 12$

Clasificación:

$f'(x) = 3x^2 - 30x + 36$

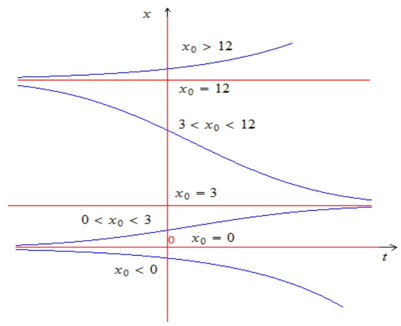
$\therefore \quad f'(0) = 36 > 0,\ f'(3) = -27 < 0,\ f'(12) = 108 > 0$

$\therefore x_1^* = 0$ es un punto de equilibrio inestable

$x_2^* = 3$ es un punto de equlibrio estable

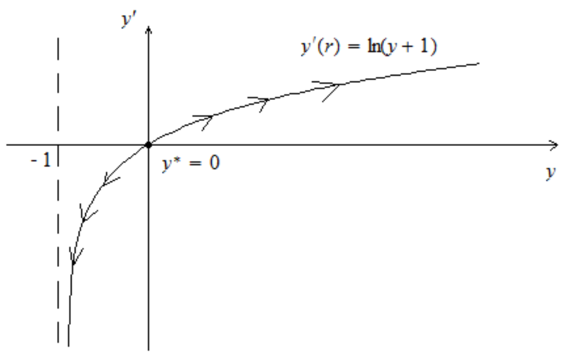
$x_3^* = 12$ es un punto de equilibrio inestable.

Gráfica de las soluciones:



(c) $y'(r) = \ln(y+1) = f(y)$

Diagrama de fase:



Puntos fijos ($y' = 0$):

$y^* = 0$

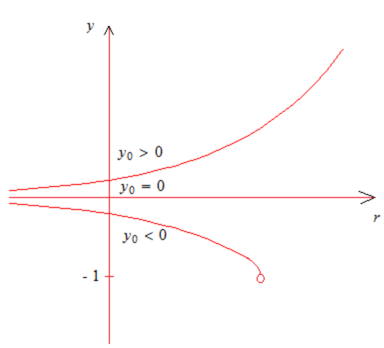
Clasificación:

$$f'(y) = \frac{1}{y+1}$$

$\therefore f'(0) = 1 > 0$

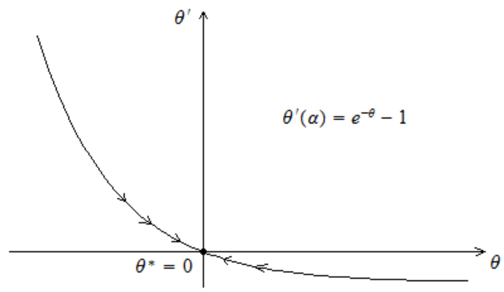
$\therefore y^* = 0$ es un punto de equilibrio inestable.

Gráfica de las soluciones:



(d) $\theta'(\alpha) = e^{-\theta} - 1 = f(\theta)$

Diagrama de fase:



Puntos fijos ($\theta' = 0$):

$\theta^* = 0$

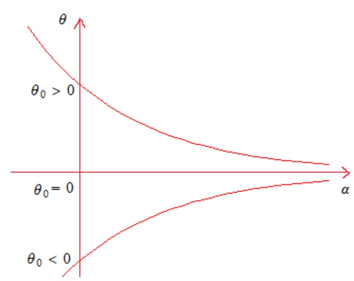
Clasificación:

$f'(\theta) = -e^{-\theta}$

$\therefore f'(0) = -1 < 0$

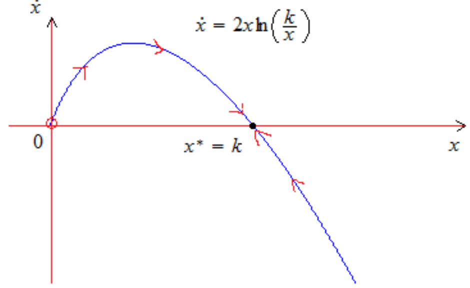
$\therefore \theta^* = 0$ es un punto de equilibrio estable.

Gráfica de las soluciones:



(e) $\dot{x} = 2x \ln\left(\dfrac{k}{x}\right) = f(x), \;\; x > 0, \, k > 0$

Diagrama de fase:



Puntos fijos ($\dot{x} = 0$):

$$\ln\left(\frac{k}{x}\right) = 0 \quad (x = 0 \text{ no está en el dominio de } f)$$

$\therefore \; x^* = k$

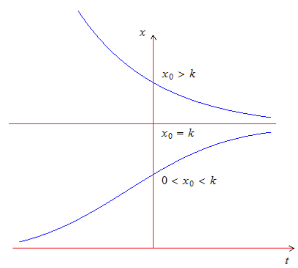
Clasificación:

$$f'(x) = 2\ln\left(\frac{k}{x}\right) + 2x\left(\frac{-1}{x}\right) = 2\left[\ln\left(\frac{k}{x}\right) - 1\right]$$

$\therefore \; f'(k) = -2 < 0$

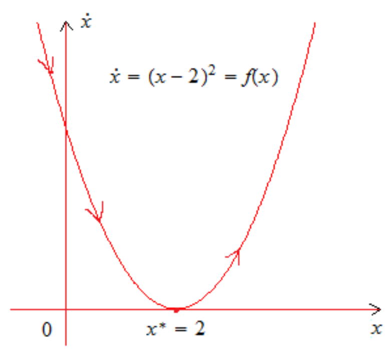
$\therefore x^* = k$ es un punto de equilibrio estable.

Gráfica de las soluciones:



(f) $\dot{x} = (x-2)^2 = f(x)$

Diagrama de fase:



Puntos fijos ($\dot{x} = 0$):

$x^* = 2$

Clasificación:

$f'(x) = 2(x-2)$

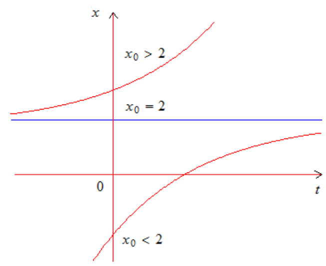
$\therefore f'(2) = 0$

$\lim_{x\to 2^+} f'(x) = 0^+$ y $\lim_{x\to 2^-} f'(x) = 0^-$

$\therefore x^* = 2$ es un punto de equilibrio estable para $x_0 < 2$,

$x^* = 2$ es un punto de equilibrio inestable para $x_0 > 2$.

Gráfica de las soluciones:



22. $\dot{x} = ax - 2a = a(x-2) = f(x)$

Puntos fijos ($\dot{x} = 0$):

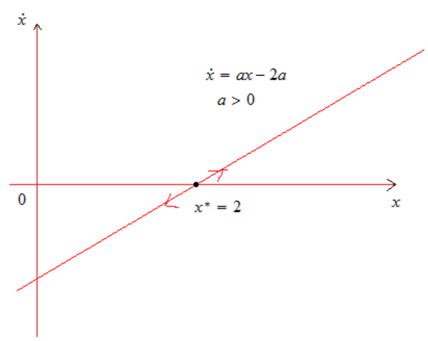
$x^* = 2$

Clasificación:

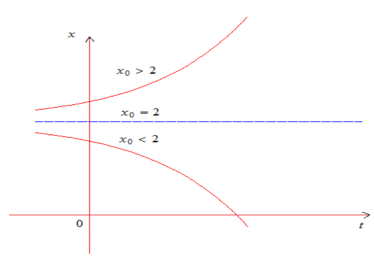
$f'(x) = a$

i. $a > 0 \Longrightarrow x^* = 2$ es un punto de equilibrio inestable.

Diagrama de fase:

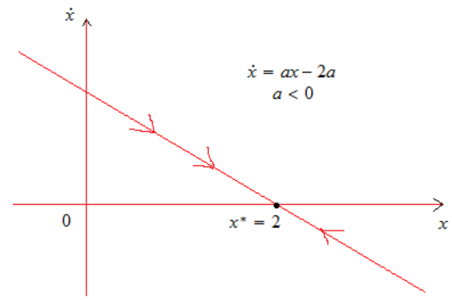


Gráfica de las soluciones:

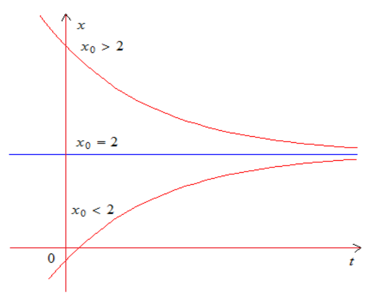


ii. $a < 0 \Longrightarrow x^* = 2$ es un punto de equilibrio estable.

(a) Diagrama de fase:



Gráfica de las soluciones:

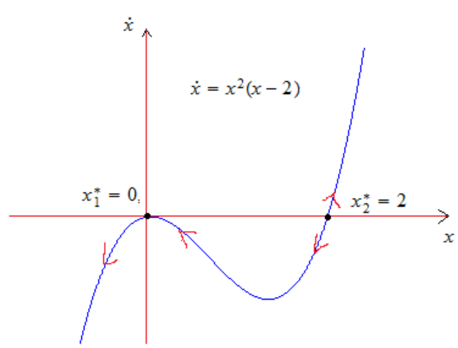


23. $\dot{x} = x^2 (x - 2) = f(x)$

(a) Puntos fijos ($\dot{x} = 0$):

$x_1^* = 0, \;\; x_2^* = 2$

(b) Diagrama de fase:

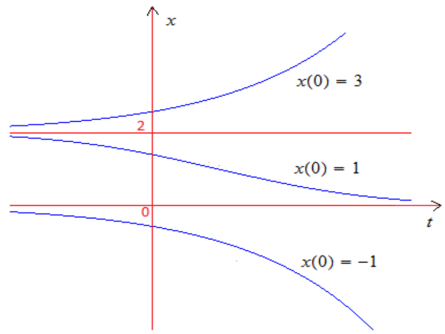


(c) $x(0) = -1 \Rightarrow \lim_{t\to\infty} x(t) = -\infty$

$x(0) = 1 \Rightarrow \lim_{t\to\infty} x(t) = 0$

$x(0) = 3 \Rightarrow \lim_{t\to\infty} x(t) = \infty$

(d) Gráfica de las soluciones:

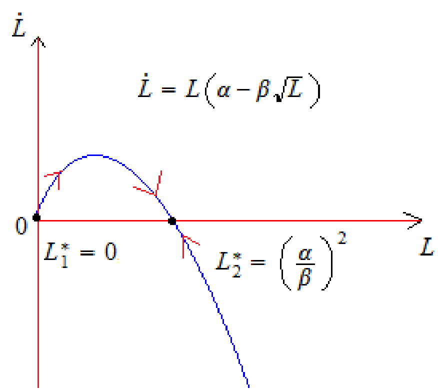


24. $\dot{L} = L\left(\alpha - \beta\sqrt{L}\right)$, $L(t) \geq 0$, $\alpha, \beta \in \Re^{+}$:

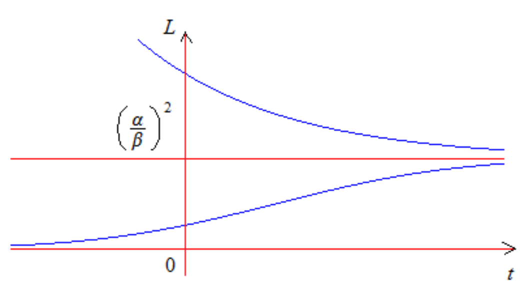
(a) Puntos fijos $\left(\dot{L} = 0\right)$:

$$L_1^* = 0, \quad L_2^* = \left(\frac{\alpha}{\beta}\right)^2$$

(b) Diagrama de fase:



(c) Gráfica de las soluciones:



(d) $\dot{L} = L\left(\alpha - \beta\sqrt{L}\right) = L\alpha - \beta L^{3/2} \longleftarrow$ Ecuación de Bernoulli

Sea $u = L^{1-3/2} = L^{-1/2}$

$$\therefore \frac{du}{dt} = -\frac{1}{2}L^{-3/2}\dot{L} = -\frac{1}{2}L^{-3/2}\left[L\alpha - \beta L^{3/2}\right]$$

$$\therefore \frac{du}{dt} = -\frac{\alpha}{2}L^{-1/2} + \frac{\beta}{2} = -\frac{\alpha}{2}u + \frac{\beta}{2}$$

$$\therefore \frac{du}{dt} + \frac{\alpha}{2}u = \frac{\beta}{2} \longleftarrow \text{Ecuación lineal autónoma}$$

$$\therefore u = Ae^{-\alpha t/2} + \frac{\beta}{\alpha}$$

$$\therefore L^{-1/2} = Ae^{-\alpha t/2} + \frac{\beta}{\alpha}$$

Solución:

$$L(t) = \left(Ae^{-\alpha t/2} + \frac{\beta}{\alpha}\right)^{-2}$$

MATEMÁTICAS APLICADAS A LA ECONOMÍA
TAREA 6 - SOLUCIONES
ECUACIONES DIFERENCIALES II
(PRIMERA PARTE)
(Tema 5.1)

1. $\ddot{x} - \dfrac{2-\alpha}{1-\alpha}a\dot{x} + \dfrac{a^2}{1-\alpha}x = 0 \qquad (\alpha \neq 0, 1 \quad a \neq 0)$

i. Sea $u_1 = e^{at}$

$\therefore\ \dot{u}_1 = ae^{at}$

$\therefore\ \ddot{u}_1 = a^2e^{at}$

De esta manera,

$$\begin{aligned}\ddot{u}_1 - \frac{2-\alpha}{1-\alpha}a\dot{u}_1 + \frac{a^2}{1-\alpha}u_1 &= a^2e^{at} - \frac{2-\alpha}{1-\alpha}a\left(ae^{at}\right) + \frac{a^2}{1-\alpha}e^{at}\\ &= a^2e^{at}\left[1 - \frac{2-\alpha}{1-\alpha} + \frac{1}{1-\alpha}\right]\\ &= a^2e^{at}\left[\frac{1-\alpha-2+\alpha+1}{1-\alpha}\right]\\ &= 0.\end{aligned}$$

ii. Sea $u_2 = e^{at/(1-\alpha)}$

$\therefore\ \dot{u}_2 = \left(\dfrac{\alpha}{1-\alpha}\right)e^{at/(1-\alpha)}$

$\therefore\ \ddot{u}_2 = \left(\dfrac{\alpha}{1-\alpha}\right)^2 e^{at/(1-\alpha)}$

De esta manera,

$$\begin{aligned}&\ddot{u}_2 - \frac{2-\alpha}{1-\alpha}a\dot{u}_2 + \frac{a^2}{1-\alpha}u_2\\ &= \left(\frac{\alpha}{1-\alpha}\right)^2 e^{at/(1-\alpha)} - \frac{2-\alpha}{1-\alpha}a\left(\frac{\alpha}{1-\alpha}\right)e^{at/(1-\alpha)} + \frac{a^2}{1-\alpha}e^{at/(1-\alpha)}\\ &= \left(\frac{\alpha}{1-\alpha}\right)^2 e^{at/(1-\alpha)}\left[1-(2-\alpha)+(1-\alpha)\right]\\ &= 0.\end{aligned}$$

Solución general:

e^{at} y $e^{at/(1-\alpha)}$ son soluciones linealmente independientes

$\therefore\ x(t) = k_1e^{at} + k_2e^{at/(1-\alpha)}, \quad k_1, k_2 \in \mathbb{R}.$

2. (a) $2\ddot{x} - 10\dot{x} + 12x = 0$

$\ddot{x} - 5\dot{x} + 6x = 0$

Proponemos $x(t) = e^{rt}$

$\therefore\ r^2 e^{rt} - 5re^{rt} + 6e^{rt} = 0$

$\therefore\ r^2 - 5r + 6 = 0$

$\therefore\ (r-2)(r-3) = 0$

$\therefore\ r_1 = 2,\ r_2 = 3$

Solución:

$x(t) = k_1 e^{2t} + k_2 e^{3t}$

(b) $\ddot{x} - x = 0, \quad x(0) = \dot{x}(0) = 1$

Proponemos $x(t) = e^{rt}$

$\therefore\ r^2 e^{rt} - e^{rt} = 0$

$\therefore\ r^2 - 1 = 0$

$\therefore\ (r+1)(r-1) = 0$

$\therefore\ r_1 = -1,\ r_2 = 1$

$\therefore\ x(t) = k_1 e^{-t} + k_2 e^{t}$

$\therefore\ \dot{x}(t) = -k_1 e^{-t} + k_2 e^{-t}$

Condición inicial:

$x(0) = 1 = k_1 + k_2$

$\dot{x}(0) = 1 = -k_1 + k_2$

$\therefore\ k_1 = 0,\ k_2 = 1$

Solución:

$x(t) = e^{t}$

(c) $\ddot{x} + 2\dot{x} + 2x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0$

Proponemos $x(t) = e^{rt}$

$\therefore\ r^2 e^{rt} + 2re^{rt} + 2e^{rt} = 0$

$\therefore\ r^2 + 2r + 2 = 0$

$\therefore\ r_{1,2} = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$\therefore\ \alpha = -1,\ \beta = 1$

$\therefore\ x(t) = e^{-t}\left[k_1 \cos t + k_2\, sen\, t\right]$

$\therefore\ \dot{x}(t) = e^{-t}\left[(k_2 - k_1)\cos t - (k_2 + k_1)\ sen\, t\right]$

Condición inicial:

$x(0) = 1 = k_1$

$\dot{x}(0) = 0 = k_2 - k_1$

$\therefore\ k_1 = 1,\ k_2 = 1$

Solución:

$x(t) = e^{-t}\left[\cos t + sen\, t\right]$

(d) $\ddot{x} - 6\dot{x} + 9x = 0, \quad x(0) = 1, \, \dot{x}(0) = 0$

Proponemos $x(t) = e^{rt}$

$\therefore \; r^2 e^{rt} - 6re^{rt} + 9e^{rt} = 0$

$\therefore \; r^2 - 6r + 9 = 0$

$\therefore \; (r-3)^2 = 0$

$\therefore \; r_1 = r_2 = 3$

$\therefore \; x(t) = k_1 e^{3t} + k_2 t e^{3t}$

$\therefore \; \dot{x}(t) = (3k_1 + k_2) e^{3t} + 3k_2 t e^{3t}$

Condición inicial:

$x(0) = 1 = k_1$

$\dot{x}(0) = 0 = 3k_1 + k_2$

$\therefore \; k_1 = 1, \, k_2 = -3$

Solución:

$x(t) = e^{3t} - 3te^{3t}$

3. (a) $\ddot{x} - \dot{x} - 2x = 4, \quad x(0) = 1, \, \dot{x}(0) = 0$

$x(t) = x_h(t) + x_p(t)$

x_h :

$\ddot{x}_h - \dot{x}_h - 2x_h = 0$

Proponemos $x_h(t) = e^{rt}$

$\therefore \; r^2 - r - 2 = 0$

$\therefore \; (r+1)(r-2) = 0$

$\therefore \; r_1 = -1, \, r_2 = 2$

$\therefore \; x_h(t) = k_1 e^{-t} + k_2 e^{2t}$

x_p :

$\ddot{x}_p - \dot{x}_p - 2x_p = 4$

Proponemos $x_p(t) = A$ (constante)

$\therefore \; A = -2$

$\therefore \; x_p(t) = -2$

$\therefore \; x(t) = k_1 e^{-t} + k_2 e^{2t} - 2$

$\therefore \; \dot{x}(t) = -k_1 e^{-t} + 2k_2 e^{2t}$

Condición inicial:

$x(0) = 1 = k_1 + k_2 - 2$

$\dot{x}(0) = 0 = -k_1 + 2k_2$

$\therefore \; k_1 = 2, \, k_2 = 1$

Solución:

$x(t) = 2e^{-t} + e^{2t} - 2$

(b) $\ddot{x} + \dot{x} = 2, \quad x(0) = 4,\ \dot{x}(0) = 1$

$x(t) = x_h(t) + x_p(t)$

$x_h :$

$\ddot{x}_h + \dot{x}_h = 0$

Proponemos $x_h(t) = e^{rt}$

$\therefore\ r^2 + r = 0$

$\therefore\ r(r+1) = 0$

$\therefore\ r_1 = -1,\ r_2 = 0$

$\therefore\ x_h(t) = k_1 e^{-t} + k_2$

$x_p :$

$\ddot{x}_p + \dot{x}_p = 2$

Proponemos $\dot{x}_p(t) = A$ (constante)

$\therefore\ x_p(t) = At$

$\therefore\ A = 2$

$\therefore\ x_p(t) = 2t$

$\therefore\ x(t) = k_1 e^{-t} + k_2 + 2t$

$\therefore\ \dot{x}(t) = -k_1 e^{-t} + 2$

Condición inicial:

$x(0) = 4 = k_1 + k_2$

$\dot{x}(0) = 1 = -k_1 + 2$

$\therefore\ k_1 = 1,\ k_2 = 3$

Solución:

$x(t) = e^{-t} + 3 + 2t$

(c) $\ddot{x} + x = 1$

$x(t) = x_h(t) + x_p(t)$

$x_h :$

$\ddot{x}_h + x_h = 0$

Proponemos $x_h(t) = e^{rt}$

$\therefore\ r^2 + 1 = 0$

$$\therefore\ r_{1,2} = \frac{\pm\sqrt{-4}}{2} = \pm i$$

$\therefore\ \alpha = 0,\ \beta = 1$

$\therefore\ x_h(t) = k_1 \cos t + k_2\, sen\, t$

$x_p :$

$\ddot{x}_p + x_p = 1$

Proponemos $x_p(t) = A$ (constante)

$\therefore\ A = 1$

$\therefore\ x_p(t) = 1$

Solución:

$x(t) = k_1 \cos t + k_2\, sen\, t + 1$

(d) $\ddot{x} + 2\dot{x} + 17x = 17$

$x(t) = x_h(t) + x_p(t)$

x_h :

$\ddot{x}_h + 2\dot{x}_h + 17x_h = 0$

Proponemos $x_h(t) = e^{rt}$

$\therefore\ r^2 + 2r + 17 = 0$

$\therefore\ r_{1,2} = \dfrac{-2 \pm \sqrt{4-68}}{2} = -1 \pm 4i$

$\therefore\ \alpha = -1,\ \ \beta = 4$

$\therefore\ x_h(t) = e^{-t}\left(k_1 \cos(4t) + k_2\, sen\ (4t)\right)$

x_p :

$\ddot{x}_p + 2\dot{x}_p + 17x_p = 17$

Proponemos $x_p(t) = A$ (constante)

$\therefore\ A = 1$

$\therefore\ x_p(t) = 1$

Solución:

$x(t) = e^{-t}\left(k_1 \cos(4t) + k_2\, sen\ (4t)\right) + 1$

4. $\ddot{x} - 4\dot{x} + 4x = f(t)$

Usaremos $x(t) = x_h(t) + x_p(t)$. Como los tres incisos tienen la misma ecuación homogénea asociada, primero resolveremos $x_h(t)$.

$\ddot{x}_h - 4\dot{x}_h + 4x_h = 0$

Proponemos $x_h(t) = e^{rt}$

$\therefore\ r^2 - 4r + 4 = 0$

$\therefore\ (r-2)^2 = 0$

$\therefore\ r_1 = r_2 = 2$

$\therefore\ x_h(t) = k_1 e^{2t} + k_2 t e^{2t}$

(a) $f(t) = 8t$

$\ddot{x}_p - 4\dot{x}_p + 4x_p = 8t$

Proponemos $x_p(t) = At + B$

$\therefore \dot{x}_p = A$

$\therefore 0 - 4A + 4(At + B) = 8t$

$\therefore 4A = 8$ y $-4A + 4B = 0$

$\therefore A = 2,\ B = 2$
$\therefore x_p\left(t\right) = 2t + 2$
Solución:
$x\left(t\right) = k_1 e^{2t} + k_2 t e^{2t} + 2t + 2$

(b) $f(t) = 4e^{-2t}$
$\ddot{x}_p - 4\dot{x}_p + 4x_p = 4e^{-2t}$
Proponemos $x_p\left(t\right) = Ae^{-2t}$
$\therefore \dot{x}_p = -2Ae^{-2t}$
$\therefore \ddot{x}_p = 4Ae^{-2t}$
$\therefore \left(4Ae^{-2t}\right) - 4\left(-2Ae^{-2t}\right) + 4Ae^{-()2t} = 4e^{-2t}$
$\therefore A = \dfrac{1}{4}$
$\therefore x_p\left(t\right) = \dfrac{1}{4}e^{-2t}$
Solución:
$x\left(t\right) = k_1 e^{2t} + k_2 t e^{2t} + \dfrac{1}{4}e^{-2t}$

(c) $f(t) = 4e^{2t}$
$\ddot{x}_p - 4\dot{x}_p + 4x_p = 4e^{2t}$
Ae^{2t} no es linealmente independiente con $k_1 e^{2t}$
Ate^{2t} no es linealmente independiente con $k_2 t e^{2t}$
$\therefore$ Proponemos $x_p\left(t\right) = At^2 e^{2t}$
$\therefore \dot{x}_p = \left(2At^2 + 2At\right) e^{2t}$
$\therefore \ddot{x}_p = \left(4At^2 + 8At + 2A\right) e^{2t}$
$\therefore \left(4At^2 + 8At + 2A\right) - 4\left(2At^2 + 2At\right) e^{2t} + 4\left(At^2 e^{2t}\right) = 4e^{2t}$
$\therefore A = 2$
$\therefore x_p\left(t\right) = 2t^2 e^{2t}$
Solución:
$x\left(t\right) = k_1 e^{2t} + k_2 t e^{2t} + 2t^2 e^{2t}$

5. (a) $y''\left(x\right) - 2y'\left(x\right) - 3y\left(x\right) = 9x^2$
$y(x) = y_h(x) + y_p(x)$
$y_h:$
$y_h'' - 2y_h' - 3y_h = 0$
Proponemos $y_h\left(x\right) = e^{rx}$
$\therefore r^2 - 2r - 3 = 0$
$\therefore \left(r + 1\right)\left(r - 3\right) = 0$
$\therefore r_1 = -1, \quad r_2 = 3$

$\therefore y_h(x) = k_1 e^{-x} + k_2 e^{3x}$

$y_p:$

$y_p'' - 2y_p' - 3y_p = 9x^2$

Proponemos $y_p(x) = Ax^2 + Bx + C$

$\therefore y_p'(x) = 2Ax + B$

$\therefore y_p''(x) = 2A$

$\therefore 2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 9x^2$

$\therefore -3A = 9,\ -4A - 3B = 0,\ 2A - 2B - 3C = 0$

$\therefore A = -3,\ B = 4,\ C = -\dfrac{14}{3}$

$\therefore y_p(x) = -3x^2 + 4x - \dfrac{14}{3}$

Solución:

$y(x) = k_1 e^{-x} + k_2 e^{3x} - 3x^2 + 4x - \dfrac{14}{3}$

(b) $\ddot{x} - 2\dot{x} - 3x = 4e^t - 9t$

$x(t) = x_h(t) + x_p(t)$

$x_h:$

$\ddot{x}_h - 2\dot{x}_h - 3x_h = 0$

Proponemos $x_h(t) = e^{rt}$

$\therefore\ r^2 - 2r - 3 = 0$

$\therefore (r+1)(r-3) = 0$

$\therefore r_1 = -1, \quad r_2 = 3$

$\therefore\ x_h(t) = k_1 e^{-t} + k_2 e^{3t}$

$x_p:$

$\ddot{x}_p - 2\dot{x}_p - 3x_p = 4e^t - 9t$

Proponemos $x_p(t) = Ae^t + Bt + C$

$\therefore \dot{x}_p(t) = Ae^t + B$

$\therefore \ddot{x}_p(t) = Ae^t$

$\therefore (Ae^t) - 2(Ae^t + B) - 3(Ae^t + Bt + C) = 4e^t - 9t$

$\therefore (-4A)e^t + (-3B)t + (-2B - 3C) = 4e^t - 9t$

$\therefore\ A = -1,\ B = 3,\ C = -2$

$\therefore\ x_p(t) = -e^t + 3t - 2$

Solución:

$x(t) = k_1 e^{-t} + k_2 e^{3t} - e^t + 3t - 2$

(c) $r''(\theta) - 2r'(\theta) - 3r(\theta) = 2e^{\theta} - 10\,sen\,\theta$

$r(\theta) = r_h(\theta) + r_p(\theta)$

r_h :

$r_h'' - 2r_h' - 3r_h = 0$

Proponemos $r_h(\theta) = e^{r\theta}$

$\therefore r^2 - 2r - 3 = 0$

$\therefore (r+1)(r-3) = 0$

$\therefore r_1 = -1, \quad r_2 = 3$

$\therefore r_h(\theta) = k_1 e^{-\theta} + k_2 e^{3\theta}$

r_p :

$r_p'' - 2r_p' - 3r_p = 2e^{\theta} - 10\,sen\,\theta$

Proponemos $r_p(\theta) = Ae^{\theta} + B\cos\theta + C\,sen\,\theta$

$\therefore \dot{r}_p(\theta) = Ae^{\theta} - B\,sen\,\theta + C\cos\theta$

$\therefore \ddot{r}_p(\theta) = Ae^{\theta} - B\cos\theta - C\,sen\,\theta$

$$\therefore \left(Ae^{\theta} - B\cos\theta - C\,sen\,\theta\right) - 2\left(Ae^{\theta} - B\,sen\,\theta + C\cos\theta\right)$$
$$-3\left(Ae^{\theta} + B\cos\theta + C\,sen\,\theta\right) = 2e^{\theta} - 10\,sen\,\theta$$

$\therefore (-4A)\,e^{\theta} + (-4B - 2C)\cos\theta + (2B - 4C)\;sen\,\theta \;=\; 2e^{\theta} - 10\,sen\,\theta$

$\therefore -4A = 2, \;\; -4B - 2C = 0, \;\; 2B - 4C = -10$

$\therefore A = -\frac{1}{2}, \; B = -1, \; C = 2$

$\therefore r_p(\theta) = -\frac{1}{2}e^{\theta} - \cos\theta + 2\,sen\,\theta$

Solución:

$$r(\theta) = k_1 e^{-\theta} + k_2 e^{3\theta} - \frac{1}{2}e^{\theta} - \cos\theta + 2\,sen\,\theta$$

(d) $x''(\alpha) - x'(\alpha) - 2x(\alpha) = 4e^{-\alpha}$

$x(\alpha) = x_h(\alpha) + x_p(\alpha)$

x_h :

$\ddot{x}_h - \dot{x}_h + 2x_h = 0$

Proponemos $x_h(\alpha) = e^{r\alpha}$

$\therefore r^2 - r + 2 = 0$

$\therefore (r+1)(r-2) = 0$

$\therefore r_1 = -1, \quad r_2 = 2$

$\therefore x_h(\alpha) = k_1 e^{-\alpha} + k_2 e^{2\alpha}$

x_p :

$\ddot{x}_p - \dot{x}_p - 2x_p = 4e^{-\alpha}$

$Ae^{-\alpha}$ no es linealmente independiente con $k_1 e^{-\alpha}$

$\therefore$ Proponemos $x_p(\alpha) = A\alpha e^{-\alpha}$

$\therefore \dot{x}_p = -A\alpha e^{-\alpha} + Ae^{-\alpha}$

$\therefore \ddot{x}_p = A\alpha e^{-\alpha} - 2Ae^{-\alpha}$

$\therefore (A\alpha e^{-\alpha} - 2Ae^{-\alpha}) - (-A\alpha e^{-\alpha} + Ae^{-\alpha}) - 2\,(A\alpha e^{-\alpha}) \;=\; 4e^{-\alpha}$

$\therefore A = -\dfrac{4}{3}$

$\therefore x_p(\alpha) = -\dfrac{4}{3}\alpha e^{-\alpha}$

Solución:

$x\,(\alpha) = k_1 e^{-\alpha} + k_2 e^{2\alpha} - \dfrac{4}{3}\alpha e^{-\alpha}$

6. $\ddot{x} - 2\dot{x} = 2e^{2t}$

i. Método de sustitución:

Sea $u = \dot{x}$

$\therefore \dot{u} - 2u = 2e^{2t}$

$\therefore \mu(t) = e^{\int -2dt} = e^{-2t}$

$\therefore \dfrac{d}{dt}\left[ue^{-2t}\right] = 2e^{2t}e^{-2t} = 2$

$\therefore ue^{-2t} = \int\; 2\; dt = 2t + C$

$\therefore u = (2t + C)\, e^{2t}$

$\therefore \dot{x} = (2t + C)\, e^{2t}$

$\therefore x = \int\; (2t + C)\, e^{2t}\; dt = te^{2t} - \dfrac{1}{2}e^{2t} + \dfrac{C}{2}e^{2t} + A$

$\therefore x = te^{2t} + Be^{2t} + A, \quad B = \dfrac{C-1}{2}$

Solución:

$x\,(t) = te^{2t} + Be^{2t} + A$

ii. Método de los coeficientes indeterminados:

$x(t) = x_h(t) + x_p(t)$

$x_h :$

$\ddot{x}_h - 2\dot{x}_h = 0$

Proponemos $x_h(t) = e^{rt}$

$\therefore r^2 - 2r = 0$

$\therefore r\,(r - 2) = 0$

$\therefore r_1 = 0, \quad r_2 = 2$

$\therefore x_h\,(t) = A + Be^{2t}$

$x_p :$

$\ddot{x}_p - 2\dot{x}_p = 2e^{2t}$

ke^{2t} no es linealmente independiente con Be^{2t}

$\therefore$ Proponemos $x_p(t) = kte^{2t}$
$\therefore \dot{x}_p = (2kt + k)\, e^{2t}$
$\therefore \ddot{x}_p = (4kt + 4k)\, e^{2t}$
$\therefore (4kt + 4k)\, e^{2t} - 2\,(2kt + k)\, e^{2t} = 2e^{2t}$
$\therefore k = 1$
$\therefore x_p(t) = te^{2t}$
Solución:
$$x\,(t) = A + Be^{2t} + te^{2t}$$

7. $\ddot{x} + 2\dot{x} + x = 8e^{-t}$

$x(t) = x_h(t) + x_p(t)$

x_h :

$\ddot{x}_h + 2\dot{x}_h + x_h = 0$

Proponemos $x_h(t) = e^{rt}$

$\therefore r^2 + 2r + 1 = 0$

$\therefore (r+1)^2 = 0$

$\therefore r_1 = r_2 = -1$

$\therefore x_h\,(t) = k_1 e^{-t} + k_2 te^{-t}$

x_p :

$\ddot{x}_p + 2\dot{x}_p + x_p = 8e^{-t}$

Ae^{-t} no es linealmente independiente con $k_1 e^{-t}$

Ate^{-t} no es linealmente independiente con $k_2 te^{-t}$

$\therefore$ Proponemos $x_p\,(t) = At^2 e^{-t}$

$\therefore \dot{x}_p\,(t) = -At^2 e^{-t} + 2Ate^{-t}$

$\therefore \ddot{x}_p\,(t) = At^2 e^{-t} - 4Ate^{-t} + 2Ae^{-t}$

$\therefore (At^2 e^{-t} - 4Ate^{-t} + 2Ae^{-t}) + 2\,(-At^2 e^{-t} + 2Ate^{-t}) + (At^2 e^{-t}) \;=\; 8e^{-t}$

$\therefore A = 4$

$\therefore x_p\,(t) = 4t^2 e^{-t}$

Solución:

$$x\,(t) = k_1 e^{-t} + k_2 te^{-t} + 4t^2 e^{-t}$$

Límite:

Sabemos que

$$\lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} \frac{1}{e^t} = 0,$$

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} \overset{\text{L'Hopital}}{=} \lim_{t\to\infty} \frac{1}{e^t} = 0,$$

$$\lim_{t\to\infty} t^2e^{-t} = \lim_{t\to\infty} \frac{t^2}{e^t} \overset{\text{L'Hopital}}{=} \lim_{t\to\infty} \frac{2t}{e^t} \overset{\text{L'Hopital}}{=} \lim_{t\to\infty} \frac{2}{e^t} = 0.$$

Por lo tanto,

$$\lim_{t\to\infty} x(t) = k_1 \lim_{t\to\infty} e^{-t} + k_2 \lim_{t\to\infty} te^{-t} + 4 \lim_{t\to\infty} t^2 e = 0.$$

A la larga, la solución converge a 0.

8. $\ddot{p} + \frac{m}{n}\dot{p} - \left(\frac{\beta+\delta}{n}\right) p = -\left(\frac{\alpha+\gamma}{n}\right), \quad \beta+\delta \neq 0, \quad m,n > 0,$
$\left(\frac{m}{n}\right)^2 + 4\frac{(\beta+\delta)}{n} = 0$

$p(t) = p_h(t) + p_p(t)$

$p_h :$

$$\ddot{p}_h + \frac{m}{n}\dot{p}_h - \left(\frac{\beta+\delta}{n}\right) p_h = 0$$

Proponemos $p_h(t) = e^{rt}$

$$\therefore r^2 + \frac{m}{n}r - \left(\frac{\beta+\delta}{n}\right) = 0$$

$$\therefore r_{1,2} = \frac{1}{2}\left[-\frac{m}{n} \pm \sqrt{\left(\frac{m}{n}\right)^2 + 4\left(\frac{\beta+\delta}{n}\right)}\right] = -\frac{m}{2n}$$

$$\therefore p_h(t) = (A+Bt)\, e^{-mt/2n}$$

$p_p :$

$$\ddot{p}_p + \frac{m}{n}\dot{p}_p - \left(\frac{\beta+\delta}{n}\right) p_p = -\left(\frac{\alpha+\gamma}{n}\right)$$

Proponemos $p_p(t) = K$

$$\therefore K = \frac{\alpha+\gamma}{\beta+\delta}$$

$$\therefore p_p(t) = \frac{\alpha+\gamma}{\beta+\delta}$$

Solución:

$$p(t) = (A+Bt)\, e^{-mt/2n} + \frac{\alpha+\gamma}{\beta+\delta}$$

Límite:

Sabemos que

$$\lim_{t\to\infty} (A+Bt)\, e^{-mt/2n} = \lim_{t\to\infty} \frac{A+Bt}{e^{mt/2n}} \overset{\text{L'Hopital}}{=} \lim_{t\to\infty} \frac{B}{\frac{m}{2n} e^{mt/2n}} = 0.$$

Por lo tanto,

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} (A+Bt)\, e^{-mt/2n} + \lim_{t\to\infty} \frac{\alpha+\gamma}{\beta+\delta} = \frac{\alpha+\gamma}{\beta+\delta}.$$

A la larga, la solución converge a $\dfrac{\alpha+\gamma}{\beta+\delta}$.

9. $\ddot{p} + \beta\dot{p} + p = m$, $\beta^2 < 4$, $m > 0$

$p(t) = p_h(t) + p_p(t)$

p_h :

$\ddot{p}_h + \beta\dot{p}_h + p_h = 0$

Proponemos $p_h(t) = e^{rt}$

$\therefore r^2 + \beta r + 1 = 0$

$\therefore r_{1,2} = \dfrac{1}{2}\left[-\beta \pm \sqrt{\beta^2 - 4}\right]$

$\therefore r_{1,2} = \dfrac{1}{2}\left[-\beta \pm i\sqrt{4 - \beta^2}\right]$ (ya que $\beta^2 < 4$)

$$\therefore p_h(t) = e^{-\beta t/2}\left[A\cos\left(\frac{\sqrt{4-\beta^2}}{2}\, t\right) + B\ sen\left(\frac{\sqrt{4-\beta^2}}{2}\, t\right)\right]$$

p_p :

$\ddot{p}_p + \beta\dot{p}_p + p_p = m$

Proponemos $p_p(t) = A$

$\therefore A = m$

$\therefore p_p(t) = m$

Solución:

$$p(t) = e^{-\beta t/2}\left[A\cos\left(\frac{\sqrt{4-\beta^2}}{2}\, t\right) + B\ sen\left(\frac{\sqrt{4-\beta^2}}{2}\, t\right)\right] + m$$

Convergencia:

Tomando en cuenta el factor $e^{-\beta t/2}$, observamos que $p(t)$ converge sólo si $\beta > 0$. Como $\beta^2 < 4$, por lo tanto la solución converge sólo si $0 < \beta < 2$. En ese caso, $\lim_{t\to\infty} p(t) = m$.

10. (a) Se tiene el sistema:

$p = \frac{1}{4} - 2U + \pi,$ ....(1)

$\pi' = \frac{1}{2}(p - \pi),$ ....(2)

$U' = p - m.$ ....(3)

De (1) se obtiene

$p - \pi = \frac{1}{4} - 2U.$ ....(4)

Sustituyendo (4) en (2) obtenemos:

$\pi' = \frac{1}{2}(\frac{1}{4} - 2U)$

$\therefore\ \pi' = \frac{1}{8} - U.$ ....(5)

Derivando ambos lados de (5) con respecto a t:

$\pi'' = -U'.$ ....(6)

Sustituyendo (3) en (6):

$\pi'' = m - p.$ ....(7)

Por otra parte, acomodando términos de (2) :

$2\pi' + \pi = p.$ ....(8)

Sustituyendo p de (8) en (7) :

$\pi'' + 2\pi' + \pi = m.$ ....(9)

(b) $\pi(t) = \pi_h(t) + \pi_p(t)$

π_h :

$\pi_h'' + 2\pi_h' + \pi_h = 0$

Proponemos $\pi_h(t) = e^{rt}$

$\therefore\ r^2 + 2r + 1 = 0$

$\therefore\ (r+1)^2 = 0$

$\therefore\ r_1 = r_2 = -1$

$\therefore \pi_h(t) = (k_1 + k_2 t)\, e^{-t}$

π_p :

$\pi_p'' + 2\pi_p' + \pi_p = m$

Proponemos $\pi_p(t) = m$

Solución:

$\pi(t) = (k_1 + k_2 t)\, e^{-t} + m$

Como $r_{1,2} \in \mathbb{R}$, la solución $\pi(t)$ no es fluctuante.

(c) Sabemos que

$$\lim_{t\to\infty} (k_1 + k_2 t)\, e^{-t} = \lim_{t\to\infty} \frac{k_1 + k_2 t}{e^t} \overset{\text{L'Hopital}}{=} \lim_{t\to\infty} \frac{k_2}{e^t} = 0.$$

Por lo tanto,

$$\lim_{t\to\infty} \pi(t) = \lim_{t\to\infty} (k_1 + k_2 t) e^{-t} + \lim_{t\to\infty} m = m.$$

$\pi(t)$ tiende a un valor constante m, de modo que ésta es estable.

11. $\ddot{\pi} + 6\dot{\pi} + 9\pi = 4e^{-3t}$

$\pi(t) = \pi_h(t) + \pi_p(t)$

π_h :

$\ddot{\pi}_h + 6\dot{\pi}_h + 9\pi_h = 0$

Proponemos $\pi_h(t) = e^{rt}$

$\therefore\ r^2 + 6r + 9 = 0$

$\therefore\ (r+3)^2 = 0$

$\therefore\ r_1 = r_2 = -3$

$\therefore\ \pi_h(t) = k_1 e^{-3t} + k_2 t e^{-3t}$

π_p :

$\ddot{\pi}_p + 6\dot{\pi}_p + 9\pi_p = 4e^{-3t}$

Ae^{-3t} no es linealmente independiente con $k_1 e^{-3t}$

Ate^{-3t} no es linealmente independiente con $k_2 t e^{-3t}$

$\therefore$ Proponemos $\pi_p(t) = At^2 e^{-3t}$

$\therefore\ \dot{\pi}_p = -3At^2 e^{-3t} + 2Ate^{-3t}$

$\therefore\ \ddot{\pi}_p = (2 - 12t + 9t^2) Ae^{-3t}$

$\therefore\ (2 - 12t + 9t^2) Ae^{-3t} + 6(-3At^2 e^{-3t} + 2Ate^{-3t}) + 9(At^2 e^{-3t}) = 4e^{-3t}$

$\therefore\ A = 2$

$\therefore\ \pi_p(t) = 2t^2 e^{-3t}$

Solución:

$\pi(t) = k_1 e^{-3t} + k_2 t e^{-3t} + 2t^2 e^{-3t}$

Límite:

Sabemos que

$$\lim_{t\to\infty} e^{-3t} = \lim_{t\to\infty} \frac{1}{e^{3t}} = 0,$$

$$\lim_{t\to\infty} te^{-3t} = \lim_{t\to\infty} \frac{t}{e^{3t}} \overset{\text{L'Hopital}}{=} \lim_{t\to\infty} \frac{1}{3e^{3t}} = 0,$$

$$\lim_{t\to\infty} t^2 e^{-3t} = \lim_{t\to\infty} \frac{t^2}{e^{3t}} \overset{\text{L'Hopital}}{=} \lim_{t\to\infty} \frac{2t}{3e^{3t}} \overset{\text{L'Hopital}}{=} \lim_{t\to\infty} \frac{2}{9e^{3t}} = 0.$$

Por lo tanto,

$$\lim_{t\to\infty} \pi(t) = k_1 \lim_{t\to\infty} e^{-3t} + k_2 \lim_{t\to\infty} te^{-3t} + 2 \lim_{t\to\infty} t^2e^{-3t} = 0.$$

A la larga, la solución converge a 0.

12. $\ddot{\pi} - 2\gamma\dot{\pi} + 4\pi = 0, \quad \gamma^2 < 4$

    Proponemos $\pi(t) = e^{rt}$

    $\therefore\ r^2 - 2\gamma r + 4 = 0$

    $\therefore\ r_{1,2} = \frac{1}{2}\left[2\gamma \pm \sqrt{4\gamma^2 - 16}\right]$

    $\therefore\ r_{1,2} = \gamma \pm \sqrt{4-\gamma^2}\ i$

    Solución:

    $$\pi(t) = e^{\gamma t}\left[A\cos\left(\sqrt{4-\gamma^2}t\right) + B\,sen\,\left(\sqrt{4-\gamma^2}t\right)\right]$$

    Convergencia:

    Tomando en cuenta el factor $e^{\gamma t}$, observamos que $\pi(t)$ converge sólo si $\gamma < 0$. Como $\gamma^2 < 4$, por lo tanto la solución converge sólo si $-2 < \gamma < 0$. En ese caso, $\lim_{t\to\infty} \pi(t) = 0$.

13. (a) $\ddot{\pi} - \dot{\pi} - 6\pi = -18, \quad \pi(0) = 5,\ \dot{\pi}(0) = K$

    $\pi(t) = \pi_h(t) + \pi_p(t)$

    π_h :

    $\ddot{\pi}_h - \dot{\pi}_h - 6\pi_h = 0$

    Proponemos $\pi_h(t) = e^{rt}$

    $\therefore\ r^2 - r - 6 = 0$

    $\therefore\ (r+2)(r-3) = 0$

    $\therefore\ r_1 = -2,\ r_2 = 3$

    $\therefore\ \pi_h(t) = Ae^{-2t} + Be^{3t}$

    π_p :

    $\ddot{\pi}_p - \dot{\pi}_p - 6\pi_p = -18$

    Proponemos $\pi_p(t) = C$ (constante)

    $\therefore\ C = 3$

    $\therefore\ \pi_p(t) = 3$

    $\therefore\ \pi(t) = Ae^{-2t} + Be^{3t} + 3$

    $\therefore\ \dot{\pi}(t) = -2Ae^{-2t} + 3Be^{3t}$

    Condiciones iniciales:

    $\pi(0) = 5 = A + B + 3$

    $\dot{\pi}(0) = K = -2A + 3B$

$\therefore\ A = \dfrac{6-K}{5},\ \ B = \dfrac{K+4}{5}$

Solución:

$\pi(t) = \left(\dfrac{6-K}{5}\right) e^{-2t} + \left(\dfrac{K+4}{5}\right) e^{3t} + 3$

(b) $\lim\limits_{t\to\infty}\ \pi(t) = \lim\limits_{t\to\infty}\ \left[\left(\dfrac{6-K}{5}\right) e^{-2t} + \left(\dfrac{K+4}{5}\right) e^{3t} + 3\right]$

Este límite converge sólo si $\dfrac{K+4}{5} = 0$, esto es, si $K = -4$.

Para $K = -4$ se tiene $\pi(t) = 2e^{-2t} + 3$.

$\therefore\ \lim\limits_{t\to\infty}\ \pi(t) = \lim\limits_{t\to\infty}\ [2e^{-2t} + 3] = 3.$

14. (a) $\dot{p}(t) = a\int_{-\infty}^{t} [D(p(\tau)) - S(p(\tau))]\, d\tau$

$\therefore\ \dfrac{d\dot{p}(t)}{dt} = a\dfrac{d}{dt}\int_{-\infty}^{t} [D(p(\tau)) - S(p(\tau))]\, d\tau\ =\ D(p(t)) - S(p(t))$

$\therefore\ \ddot{p}(t) = a[D(p(t)) - S(p(t))]$

(b) $D(p) = d_0 + d_1 p,\ S(p) = s_0 + s_1 p,\ \ d_1 < 0,\ s_1 > 0$

$\ddot{p}(t) = a[D(p(t)) - S(p(t))]$

$\therefore\ \ddot{p} = a[(d_0 + d_1 p) - (s_0 + s_1 p)]$

$\therefore\ \ddot{p} = a(d_1 - s_1)p + a(d_0 - s_0)$

$\therefore\ \ddot{p} - a(d_1 - s_1)p = a(d_0 - s_0)$

$p(t) = p_h(t) + p_p(t)$

$p_h:$

$\ddot{p}_h - a(d_1 - s_1)p_h = 0$

Proponemos $p_h(t) = e^{rt}$

$\therefore\ r^2 - a(d_1 - s_1) = 0$

$\therefore\ r^2 = a(d_1 - s_1) < 0$

$\therefore\ r_{1,2} = \pm\sqrt{a(d_1 - s_1)} = \pm\sqrt{a(s_1 - d_1)}\ i$

$\therefore\ \alpha = 0,\ \ \beta = \sqrt{a(s_1 - d_1)}$

$\therefore\ p_h(t) = A\cos\left(\sqrt{a(s_1 - d_1)}t\right) + B\, sen\ \left(\sqrt{a(s_1 - d_1)}t\right)$

$p_p:$

$\ddot{p}_p - a(d_1 - s_1)p_p = a(d_0 - s_0)$

Proponemos $p_p(t) = K$ (constante)

$\therefore\ K = -\left(\dfrac{d_0 - s_0}{d_1 - s_1}\right)$

$\therefore\ p_p(t) = -\left(\dfrac{d_0 - s_0}{d_1 - s_1}\right)$

Solución:

$$p\left(t\right)=A\cos\left(\sqrt{a\left(s_1-d_1\right)}t\right)+B\,sen\,\left(\sqrt{a\left(s_1-d_1\right)}t\right)\;-\;\left(\frac{d_0-s_0}{d_1-s_1}\right)$$

15. $\ddot{p}(t)=\gamma(\beta-\alpha)p(t)+k,\ \gamma,\alpha,a,k$ constantes

Casos:

(a) $\gamma(\beta-\alpha)=0$

$\ddot{p}(t)=k$

$\therefore\ \dot{p}\left(t\right)=kt+A$

$\therefore\ p\left(t\right)=\dfrac{k}{2}t^2+At+B$

(b) $\gamma(\beta-\alpha)\neq 0$

$p\left(t\right)=p_h\left(t\right)+p_p\left(t\right)$

$p_h:$

$\ddot{p}(t)-\gamma(\beta-\alpha)p(t)=0,$

Proponemos $p_h\left(t\right)=e^{rt}$

$\therefore\ r^2-\gamma(\beta-\alpha)=0$

$\therefore\ r_{1,2}=\pm\sqrt{\gamma(\beta-\alpha)}$

Casos:

b.1 $\gamma(\beta-\alpha)>0$

$r_{1,2}=\pm\sqrt{\gamma(\beta-\alpha)}\in\mathbb{R}\longleftarrow$raíces reales distintas

$\therefore\ p_h\left(t\right)=Ae^{\sqrt{\gamma(\beta-\alpha)}t}+Be^{-\sqrt{\gamma(\beta-\alpha)}}$

b.2 $\gamma(\beta-\alpha)<0$

$r_{1,2}=\pm\sqrt{-\gamma(\alpha-\beta)}=\pm\sqrt{\gamma(\alpha-\beta)}\ i$

$\therefore\ p_h\left(t\right)=A\cos\left(\sqrt{\gamma(\alpha-\beta)}t\right)+B\,sen\,\left(\sqrt{\gamma(\alpha-\beta)}t\right)$

$p_p:$

$\ddot{p}_p(t)-\gamma(\beta-\alpha)p_p(t)=k$

Proponemos $p_p\left(t\right)=C$ (constante)

$\therefore\ C=-\left(\dfrac{k}{\gamma(\beta-\alpha)}\right)=\dfrac{k}{\gamma(\alpha-\beta)}$

$\therefore\ p_p\left(t\right)=\dfrac{k}{\gamma(\alpha-\beta)}$

Solución:

$$p(t)=\begin{cases}Ae^{\sqrt{\gamma(\beta-\alpha)}t}+Be^{-\sqrt{\gamma(\beta-\alpha)}t}+\dfrac{k}{\gamma(\alpha-\beta)}, & \gamma(\beta-\alpha)>0\\ \dfrac{k}{2}t^2+At+B, & \gamma(\beta-\alpha)=0\\ A\cos\left(\sqrt{\gamma(\alpha-\beta)}t\right)+B\,sen\,\left(\sqrt{\gamma(\alpha-\beta)}t\right)+\dfrac{k}{\gamma(\alpha-\beta)}, & \gamma(\beta-\alpha)<0\end{cases}$$

16. (a) $\dot{x} = 8e^{2t}\int_0^t e^{-2u}x(u)du, \;\; x(0) = 3$

Primero observa que

$$\dot{x}(0) = 8e^0 \int_0^0 e^{-2u}x(u)\,du = 0.$$

Por otra parte,

$$\ddot{x} = \left(\frac{d\ 8e^{2t}}{dt}\right)\int_0^t e^{-2u}x(u)\,du + 8e^{2t}\left(\frac{d}{dt}\int_0^t e^{-2u}x(u)\,du\right)$$

$$\therefore\ \ddot{x} = \underbrace{2\left(8e^{2t}\right)\int_0^t e^{-2u}x(u)\,du}_{\dot{x}} + 8e^{2t}\left[e^{-2t}x(t)\right]$$

$\therefore\ \ddot{x} = 2\dot{x} + 8x$

$\therefore\ \ddot{x} - 2\dot{x} - 8x = 0$

Proponemos $x(t) = e^{rt}$

$\therefore\ r^2 - 2r - 8 = 0$

$\therefore\ (r+2)(r-4) = 0$

$\therefore\ r_1 = -2, \;\; r_2 = 4$

$\therefore\ x(t) = k_1e^{-2t} + k_2e^{4t}$

$\therefore\ \dot{x}(t) = -2k_1e^{-2t} + 4k_2e^{4t}$

Condiciones iniciales:

$x(0) = 3 = k_1 + k_2$

$\dot{x}(0) = 0 = -2k_1 + 4k_2$

$\therefore\ k_1 = 2, \;\; k_2 = 1$

Solución:

$x(t) = 2e^{-2t} + e^{4t}$

(b) $\dot{x} = 4x + e^{4t}\int_0^t e^{-4u}x(u)du, \;\; x(0) = 8$

Primero observa que

$$\dot{x}(0) = 4x(0) + e^0\int_0^0 e^{-4u}x(u)du = 4(8) = 32.$$

Por otra parte,

$$\ddot{x} = 4\dot{x} + \left(\frac{d\ e^{4t}}{dt}\right)\int_0^t e^{-4u}x(u)\,du + e^{4t}\left(\frac{d}{dt}\int_0^t e^{-4u}x(u)\,du\right)$$

$$\therefore\ \ddot{x} = 4\dot{x} + \left(4e^{4t}\right)\int_0^t e^{-4u}x(u)\,du + e^{4t}\left(e^{-4t}x(t)\right)$$

$$\therefore\ \ddot{x} = 4\dot{x} + 4\underbrace{\left(e^{4t}\right)\int_0^t e^{-4u} x\,(u)\,du}_{=\dot{x}-4x} + x$$

$\therefore\ \ddot{x} = 4\dot{x} + 4\,(\dot{x} - 4x) + x$

$\therefore\ \ddot{x} - 8\dot{x} + 15x = 0$

Proponemos $x\,(t) = e^{rt}$

$\therefore\ r^2 - 8r + 15 = 0$

$\therefore\ (r-3)\,(r-5) = 0$

$\therefore\ r_1 = 3,\ \ r_2 = 5$

$\therefore\ x\,(t) = k_1 e^{3t} + k_2 e^{5t}$

$\therefore\ \dot{x}\,(t) = 3k_1 e^{3t} + 5k_2 e^{5t}$

Condiciones iniciales:

$x(0) = 8 = k_1 + k_2$

$\dot{x}\,(0) = 32 = 3k_1 + 5k_2$

$\therefore\ k_1 = k_2 = 4$

Solución:

$x\,(t) = 4e^{3t} + 4e^{5t}$

17. (a) $x(t) = Ae^{2t} + Be^{t}$

$x(t)$ es de la forma $x(t) = \underbrace{Ae^{r_1 t} + Be^{r_2 t}}_{x_h(t)}$

$\therefore\ x(t)$ es solución a la ecuación homogénea $a\ddot{x} + b\dot{x} + cx = 0$

Para encontrar las constantes a, b, c observamos que

$r_1 = 2, \quad r_2 = 1$

$\therefore\ (r-2)\,(r-1) = 0$

$\therefore\ r^2 - 3r + 2 = 0$

$\therefore\ a = 1,\ \ b = -3,\ \ c = 2$

Ecuación:

$\ddot{x} - 3\dot{x} + 2x = 0$

(b) $x(t) = e^{-2t}(A\cos t + B\,sen\,t)$

$x(t)$ es de la forma $x(t) = \underbrace{e^{\alpha t}\left(A\cos\left(\beta t\right) + B\,sen\,\left(\beta t\right)\right)}_{x_h(t)}$

$\therefore\ x(t)$ es solución a la ecuación homogénea $a\ddot{x} + b\dot{x} + cx = 0$

Para encontrar las constantes a, b, c observamos que

$r_{1,2} = \alpha \pm i\beta$, con $\alpha = -2$, $\beta = 1$

$\therefore\ r_{1,2} = -2 \pm i$

$\therefore\ (r-(-2+i))\,(r-(-2-i))=0$

$\therefore\ r^2-r\,(-2-i)-r\,(-2+i)+(-2+i)\,(-2-i)=0$

$\therefore\ r^2+2r+ri+2r-ri+4-i^2=r^2+4r+5=0$

$\therefore\ r^2+4r+5=0$

$\therefore\ a=1,\ \ b=4,\ \ c=5$

Ecuación:

$\ddot{x}+4\dot{x}+5x=0$

(c) $x(t)=Ae^{2t}+Be^{t}+3t+1$

$x(t)$ es de la forma $x(t)=\underbrace{Ae^{r_1t}+Be^{r_2t}}_{x_h(t)}+\underbrace{3t+1}_{x_p(t)}$

$\therefore\ x(t)$ es solución a la ecuación no homogénea $a\ddot{x}+b\dot{x}+cx\ =\ f(t)$

i. Para encontrar las constantes a,b,c observamos que

$r_1=2,\quad r_2=1$

$\therefore\ (r-2)\,(r-1)=0$

$\therefore\ r^2-3r+2=0$

$\therefore\ a=1,\ \ b=-3,\ \ c=2$

$\therefore$ la ecuación homogénea asociada es

$\ddot{x}_h-3\dot{x}_h+2x_h=0$

ii. Para encontrar la función $f(t)$ observamos que $x_p(t)\ =\ 3t+1$ es una solución particular a la ecuación no homogénea, esto es:

$\ddot{x}_p-3\dot{x}_p+2x_p=f\,(t)$

$\therefore\ 0-3\,(3)+2\,(3t+1)=f\,(t)$

$\therefore\ f\,(t)=6t-7$

Ecuación:

$\ddot{x}-3\dot{x}+2x=6t-7$

18. $\dddot{x}-2\ddot{x}-\dot{x}+2x=0$

Proponemos $x\,(t)=e^{rt}$

$r^3-2r^2-r+2=0$

$(r-1)\,(r^2-r-2)=0$

$(r-1)\,(r-2)\,(r+1)=0$

$r_1=1,\ \ r_2=2,\ \ r_3=-1$

Solución:

$x\,(t)=k_1e^{t}+k_2e^{2t}+k_3e^{-t}$

19. $\ddot{k} = (\gamma_1\lambda + \gamma_2)\dot{k} + (\gamma_1\sigma + \gamma_3)\mu_0 e^{\mu t}\int_0^t e^{-\mu\tau}\dot{k}(\tau)d\tau,$
$\gamma_1, \gamma_2, \gamma_3, \lambda, \sigma, \mu_0, \mu$ son constantes

Por simplicidad, definamos

$\alpha = \gamma_1\lambda + \gamma_2$

$\beta = (\gamma_1\sigma + \gamma_3)\mu_0$

Por lo tanto, la ecuación se convierte en

$$\ddot{k} = \alpha\dot{k} + \beta e^{\mu t}\int_0^t e^{-\mu\tau}\dot{k}(\tau)\,d\tau.$$

Derivamos respecto a t:

$$\dddot{k} = \alpha\ddot{k} + \left(\frac{d\ \beta e^{\mu t}}{dt}\right)\int_0^t e^{-\mu\tau}\dot{k}(\tau)\,d\tau + \beta e^{\mu t}\left(\frac{d}{dt}\int_0^t e^{-\mu\tau}\dot{k}(\tau)\,d\tau\right)$$

$$\therefore\ \dddot{k} = \alpha\ddot{k} + (\beta\mu e^{\mu t})\int_0^t e^{-\mu\tau}\dot{k}(\tau)\,d\tau + \beta e^{\mu t}\left(e^{-\mu t}\dot{k}(t)\right)$$

$$\therefore\ \dddot{k} = \alpha\ddot{k} + \underbrace{\mu(\beta e^{\mu\tau})\int_0^t e^{-\mu\tau}\dot{k}(\tau)\,d\tau}_{=\ddot{k}-\alpha\dot{k}} + \beta\dot{k}$$

$$\therefore\ \dddot{k} = \alpha\ddot{k} + \mu\left(\ddot{k} - \alpha\dot{k}\right) + \beta\dot{k}$$

$$\therefore\ \dddot{k} - (\alpha+\mu)\ddot{k} + (\alpha\mu - \beta)\dot{k} = 0$$

Proponemos $k(t) = e^{rt}$

$\therefore\ r^3 - (\alpha+\mu)r^2 + (\alpha\mu - \beta)r = 0$

$\therefore\ r\left(r^2 - (\alpha+\mu)r + (\alpha\mu - \beta)\right) = 0$

$\therefore\ r_3 = 0$

Las otras dos raíces son

$$r_{1,2} = \frac{(\alpha+\mu) \pm \sqrt{(\alpha+\mu)^2 - 4(\alpha\mu - \beta)}}{2}$$

r_1 y r_2 son reales y distintas si y sólo si $(\alpha-\mu)^2 - 4(\alpha\mu - \beta) > 0$

En este caso

$k(t) = C_1e^{r_1t} + C_2e^{r_2t} + C_3e^0$

Solución:

$k(t) = C_1e^{r_1t} + C_2e^{r_2t} + C_3$

MATEMÁTICAS APLICADAS A LA ECONOMÍA
TAREA 7 - SOLUCIONES
ECUACIONES DIFERENCIALES II
(SEGUNDA PARTE)
(Temas 5.2-5.5)

1. (a) $\dot{\overrightarrow{X}} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \overrightarrow{X}, \qquad A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

$P_A(\lambda) = \lambda^2 + 2\lambda + 5 = 0$

$\therefore\ \lambda_{1,2} = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$\therefore\ \alpha = -1,\ \beta = 2$

$\lambda_1 = -1 + 2i:$

$$\begin{pmatrix} -1-(-1+2i) & 2 \\ -2 & -1-(-1+2i) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ -2im + 2n = 0$

$\therefore\ n = im$

$m = 1 \Rightarrow n = i \quad \therefore \vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{r} + i\vec{s}$

$\therefore\ \vec{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \vec{s} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solución:

$$\overrightarrow{X}(t) = k_1 e^{-t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} sen\ (2t) \right] + k_2 e^{-t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} sen\ (2t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) \right]$$

(b) $\dot{\overrightarrow{X}} = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix} \overrightarrow{X}, \qquad A = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix}$

$P_A(\lambda) = \lambda^2 + 4 = 0$

$\therefore\ \lambda_{1,2} = \dfrac{\pm\sqrt{-16}}{2} = \pm 2i$

$\therefore\ \alpha = 0,\ \beta = 2$

$\lambda_1 = 2i:$

$$\begin{pmatrix} -1-2i & 5 \\ -1 & 1-2i \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ (-1-2i)\, m + 5n = 0$

$\therefore\ n = \dfrac{1}{5} m (1+2i)$

$$m = 5 \Rightarrow n = 1 + 2i$$

$$\therefore\ \vec{v}_1 = \begin{pmatrix} 5 \\ 1+2i \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \vec{r} + i\vec{s}$$

$$\therefore\ \vec{r} = \begin{pmatrix} 5 \\ 1 \end{pmatrix},\ \vec{s} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Solución:

$$\overrightarrow{X}(t) = k_1 \left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} sen(2t) \right] + k_2 \left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} sen(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right]$$

(c) $\dot{\overrightarrow{X}} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \overrightarrow{X} + \begin{pmatrix} -4 \\ 4 \end{pmatrix}$

$\overrightarrow{X}(t) = \overrightarrow{X}_h(t) + \overrightarrow{X}_p(t)$

$\overrightarrow{X}_h$:

$$\dot{\overrightarrow{X}}_h = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \overrightarrow{X}_h, \qquad A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$

$$P_A(\lambda) = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2) = 0$$

$\therefore\ \lambda_1 = 2,\ \ \lambda_2 = -2$

$\lambda_1 = 2$:

$$\begin{pmatrix} 1-2 & 3 \\ 1 & -1-2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ -m + 3n = 0$$

$$\therefore\ m = 3n$$

$$n = 1 \Rightarrow m = 3 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$\lambda_2 = -2$:

$$\begin{pmatrix} 1-(-2) & 3 \\ 1 & -1-(-2) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ m + n = 0$$

$$\therefore\ n = -m$$

$$m = 1 \Rightarrow n = -1 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore\ \overrightarrow{X}_h(t) = k_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

$\overrightarrow{X}_p$:

$$\dot{\overrightarrow{X}}_p = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \overrightarrow{X}_p + \begin{pmatrix} -4 \\ 4 \end{pmatrix}$$

Proponemos $\overrightarrow{X}_p(t) = \begin{pmatrix} A \\ B \end{pmatrix} \quad \therefore \quad \dot{\overrightarrow{X}}_p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} -4 \\ 4 \end{pmatrix}$$

$\therefore\ 0 = A + 3B - 4$

$0 = A - B + 4 \qquad \therefore \qquad A = -2, \quad B = 2$

$$\therefore\ \overrightarrow{X}_p(t) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

(d) $\dot{\overrightarrow{X}} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \overrightarrow{X}, \qquad A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$P_A(\lambda) = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0$

$\therefore\ \lambda_1 = 2,\ \lambda_2 = 4$

$\lambda_1 = 2:$

$$\begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ m + n = 0$

$\therefore\ n = -m$

$m = 1 \Rightarrow n = -1 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 4:$

$$\begin{pmatrix} 3-4 & 1 \\ 1 & 3-4 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ -m + n = 0$

$\therefore\ n = m$

$m = 1 \Rightarrow n = 1 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

(e) $\dot{\overrightarrow{X}} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \overrightarrow{X},, \qquad A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

$$P_A(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

$\therefore\ \lambda_1 = \lambda_2 = 3$

$\lambda_1 = 3:$

$$\begin{pmatrix} 4-3 & -1 \\ 1 & 2-3 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ m - n = 0$

$\therefore\ n = m$

$$m = 1 \Rightarrow n = 1 \quad \therefore \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\vec{w}:$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore\ w_1 - w_2 = 1$

$\therefore\ w_2 = w_1 - 1$

$$w_1 = 2 \Rightarrow w_2 = 1 \quad \therefore \quad \vec{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + k_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] e^{3t}$$

(f) $\dot{\overrightarrow{X}} = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} \overrightarrow{X}, \qquad A = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix}$

$$P_A(\lambda) = \lambda^2 + 5\lambda = \lambda(\lambda + 5) = 0$$

$\therefore\ \lambda_1 = 0,\ \ \lambda_2 = -5$

$\lambda_1 = 0:$

$$\begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ m + 3n = 0$

$\therefore\ m = -3n$

$$n = 1 \Rightarrow m = -3 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$\lambda_2 = -5:$

$$\begin{pmatrix} 1-(-5) & 3 \\ -2 & -6-(-5) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 6 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ -2m - n = 0$
$\therefore\ n = -2m$

$$m = 1 \Rightarrow n = -2 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t}$$

2. (a) $\dot{\overrightarrow{X}} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \overrightarrow{X} + \begin{pmatrix} 2e^{-2t} \\ 32e^{-2t} \end{pmatrix}$

$\overrightarrow{X}(t) = \overrightarrow{X}_h(t) + \overrightarrow{X}_p(t)$

$\overrightarrow{X}_h :$

$$\dot{\overrightarrow{X}}_h = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \overrightarrow{X}_h, \qquad A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$$P_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

$\therefore\ \lambda_1 = 2,\ \ \lambda_2 = 3$

$\lambda_1 = 2 :$

$$\begin{pmatrix} 1-2 & 1 \\ -2 & 4-2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ -m + n = 0$
$\therefore\ n = m$

$$m = 1 \Rightarrow n = 1 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 3 :$

$$\begin{pmatrix} 1-3 & 1 \\ -2 & 4-3 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ -2m + n = 0$
$\therefore\ n = 2m$

$$m = 1 \Rightarrow n = 2 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore\ \overrightarrow{X}_h(t) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

$\overrightarrow{X}_p$:

$$\dot{\overrightarrow{X}}_p = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \overrightarrow{X}_p + \begin{pmatrix} 2 \\ 32 \end{pmatrix} e^{-2t}$$

Proponemos $\overrightarrow{X}_p(t) = \begin{pmatrix} A \\ B \end{pmatrix} e^{-2t} \quad \therefore \quad \dot{\overrightarrow{X}}_p = \begin{pmatrix} -2A \\ -2B \end{pmatrix} e^{-2t}$

$\therefore \begin{pmatrix} -2Ae^{-2t} \\ -2Be^{-2t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} Ae^{-2t} \\ Be^{-2t} \end{pmatrix} + \begin{pmatrix} 2e^{-2t} \\ 32e^{-2t} \end{pmatrix}$

$\therefore -2A = A + B + 2$

$-2B = -2A + 4B + 32$

$\therefore A = 1, \quad B = -5$

$\therefore \overrightarrow{X}_p(t) = \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-2t}$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-2t}$$

(b) $\dot{\overrightarrow{X}} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \overrightarrow{X} + \begin{pmatrix} -3t \\ 11 \end{pmatrix}$

$\overrightarrow{X}(t) = \overrightarrow{X}_h(t) + \overrightarrow{X}_p(t)$

$\overrightarrow{X}_h(t) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$ (ver inciso anterior)

$\overrightarrow{X}_p$:

$\dot{\overrightarrow{X}}_p = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \overrightarrow{X}_p + \begin{pmatrix} -3 \\ 11 \end{pmatrix} t$

Proponemos $\overrightarrow{X}_p(t) = \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} C \\ D \end{pmatrix} t \quad \therefore \quad \dot{\overrightarrow{X}}_p = \begin{pmatrix} C \\ D \end{pmatrix}$

$\therefore \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} A + Ct \\ B + Dt \end{pmatrix} + \begin{pmatrix} -3t \\ 11 \end{pmatrix}$

$\therefore \underbrace{(A + B - C)}_{0} + \underbrace{(C + D - 3)}_{0} t = 0$

$\underbrace{(-2A + 4B + 11 - D)}_{0} + \underbrace{(-2C + 4D)}_{0} t = 0$

$\therefore A = 3, \quad B = -1, \quad C = 2, \quad D = 1$

$\therefore \overrightarrow{X}_p(t) = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} t$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} t$$

(c) $\dot{\overrightarrow{X}} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \overrightarrow{X} + \begin{pmatrix} 0 \\ 4e^{-t} \end{pmatrix}, \quad \overrightarrow{X}(0) = \overrightarrow{0}$

$\overrightarrow{X}(t) = \overrightarrow{X}_h(t) + \overrightarrow{X}_p(t)$

$\overrightarrow{X}_h$:

$$\dot{\overrightarrow{X}}_h = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \overrightarrow{X}_h, \qquad A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$P_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

$\therefore\ \lambda_1 = 2,\ \ \lambda_2 = 3$

$\lambda_1 = 2$:

$$\begin{pmatrix} 2-2 & 1 \\ 0 & 3-2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ n = 0\ \ (m \text{ es libre})$

$$m = 1 \Rightarrow n = 0 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = 3$:

$$\begin{pmatrix} 2-3 & 1 \\ 0 & 3-3 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ -m + n = 0$

$\therefore\ n = m$

$$m = 1 \Rightarrow n = 1 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore\ \overrightarrow{X}_h(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$\overrightarrow{X}_p$:

$$\dot{\overrightarrow{X}}_p = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \overrightarrow{X}_p + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t}$$

Proponemos $\overrightarrow{X}_p(t) = \begin{pmatrix} A \\ B \end{pmatrix} e^{-t} \quad \therefore \quad \dot{\overrightarrow{X}}_p = \begin{pmatrix} -A \\ -B \end{pmatrix} e^{-t}$

$$\therefore\ \begin{pmatrix} -Ae^{-t} \\ -Be^{-t} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} Ae^{-t} \\ Be^{-t} \end{pmatrix} + \begin{pmatrix} 0 \\ 4e^{-t} \end{pmatrix}$$

$\therefore\ -A = 2A + B$

$-B = 3B + 4$

$\therefore\ A = \dfrac{1}{3}, \quad B = -1$

$$\therefore\ \overrightarrow{X}_p(t) = \begin{pmatrix} \frac{1}{3} \\ -1 \end{pmatrix} e^{-t} = \frac{1}{3} \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-t}$$

$$\therefore\ \overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \frac{1}{3} \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-t}$$

Condición inicial:

$$\overrightarrow{X}(0)=\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}k_1+k_2+\frac{1}{3}\\k_2-1\end{pmatrix}$$

$\therefore\ k_1=-\frac{4}{3},\quad k_2=1$

Solución:

$$\overrightarrow{X}(t)=-\frac{4}{3}\begin{pmatrix}1\\0\end{pmatrix}e^{2t}+\begin{pmatrix}1\\1\end{pmatrix}e^{3t}+\frac{1}{3}\begin{pmatrix}1\\-3\end{pmatrix}e^{-t}$$

(d) $\dot{x}=x-t$

$\dot{y}=2x-y$

Se tiene $\dot{\overrightarrow{X}}=\begin{pmatrix}1&0\\2&-1\end{pmatrix}\overrightarrow{X}-\begin{pmatrix}t\\0\end{pmatrix}$

$\overrightarrow{X}(t)=\overrightarrow{X}_h(t)+\overrightarrow{X}_p(t)$

$\overrightarrow{X}_h$:

$$\dot{\overrightarrow{X}}_h=\begin{pmatrix}1&0\\2&-1\end{pmatrix}\overrightarrow{X}_h,\qquad A=\begin{pmatrix}1&0\\2&-1\end{pmatrix}$$

$$P_A(\lambda)=\lambda^2-1=(\lambda+1)(\lambda-1)=0$$

$\therefore\ \lambda_1=-1,\ \lambda_2=1$

$\lambda_1=-1$:

$$\begin{pmatrix}1-(-1)&0\\2&-1-(-1)\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}2&0\\2&0\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\therefore\ m=0\ \ (n$ es libre)

$$n=1\Rightarrow m=0\quad\therefore\quad\vec{v}_1=\begin{pmatrix}0\\1\end{pmatrix}$$

$\lambda_2=1$:

$$\begin{pmatrix}1-1&0\\2&-1-1\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}0&0\\2&-2\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\therefore\ 2m-2n=0$

$\therefore\ n=m$

$$m=1\Rightarrow n=1\quad\therefore\quad\vec{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\therefore\ \overrightarrow{X}_h(t)=k_1\begin{pmatrix}0\\1\end{pmatrix}e^{-t}+k_2\begin{pmatrix}1\\1\end{pmatrix}e^{t}$$

$\overrightarrow{X}_p$:

$$\dot{\overrightarrow{X}}_p = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \overrightarrow{X}_p + \begin{pmatrix} -1 \\ 0 \end{pmatrix} t$$

Proponemos $\overrightarrow{X}_p(t) = \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} C \\ D \end{pmatrix} t \quad \therefore \quad \dot{\overrightarrow{X}}_p = \begin{pmatrix} C \\ D \end{pmatrix}$

$$\therefore \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} A + Ct \\ B + Dt \end{pmatrix} - \begin{pmatrix} t \\ 0 \end{pmatrix}$$

$$\therefore \; C = A + Ct - t$$
$$D = 2A + 2Ct - B - Dt$$

$$\therefore \; \underbrace{(A - C)}_{0} + \underbrace{(C - 1)}_{0} t = 0$$
$$\underbrace{(2A - B - D)}_{0} + \underbrace{(2C - D)}_{0} t = 0$$

$$\therefore \; A = 1, \quad B = 0, \quad C = 1, \quad D = 2$$

$$\therefore \; \overrightarrow{X}_p(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t$$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t$$

3. (a) $\overrightarrow{X}(t) = k_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$

$\overrightarrow{X}(t)$ es de la forma $\overrightarrow{X}(t) = \underbrace{k_1 \overrightarrow{v}_1 e^{\lambda_1 t} + k_2 \overrightarrow{v}_2 e^{\lambda_2 t}}_{\overrightarrow{X}_h(t)}$

$\therefore \; \overrightarrow{X}(t)$ es solución a la ecuación homogénea $\dot{\overrightarrow{X}} = A\overrightarrow{X}$,

con $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Para encontrar la matriz A observamos que sus valores y vectores propios son

$$\overrightarrow{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \; \lambda_1 = -1; \; \overrightarrow{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \; \lambda_2 = 3$$

Por lo tanto, se cumple:

$$A\overrightarrow{v}_1 = \lambda_1 \overrightarrow{v}_1$$
$$A\overrightarrow{v}_2 = \lambda_2 \overrightarrow{v}_2,$$

esto es,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Se tiene:

$$\begin{matrix} 2a-b=-2 \\ 2c-d=1 \end{matrix} \quad \text{y} \quad \begin{matrix} 2a+b=6 \\ 2c+d=3 \end{matrix},$$

de donde $a=1$, $b=4$, $c=1$ y $d=1$.

Por lo tanto,

$$A=\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$$

Ecuación:

$$\dot{\overrightarrow{X}}=\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\overrightarrow{X}$$

(b) $\overrightarrow{X}(t)=k_1\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^t+k_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}+\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\overrightarrow{X}(t)$ es de la forma $\overrightarrow{X}(t)=\underbrace{k_1\overrightarrow{v}_1e^{\lambda_1 t}+k_2\overrightarrow{v}_2e^{\lambda_2 t}}_{\overrightarrow{X}_h(t)}+\underbrace{\overrightarrow{B}}_{\overrightarrow{X}_p(t)}$

$\therefore$ $\overrightarrow{X}(t)$ es solución a la ecuación no homogénea $\dot{\overrightarrow{X}}=A\overrightarrow{X}+\overrightarrow{B}$, con $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ y $\overrightarrow{B}=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

i. Para encontrar la matriz A observamos que sus valores y vectores propios son

$$\overrightarrow{v}_1=\begin{pmatrix} 3 \\ -1 \end{pmatrix},\ \lambda_1=1;\ \overrightarrow{v}_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \lambda_2=-1$$

Por lo tanto, se cumple:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}=1\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}=-1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Resolviendo el sistema de ecuaciones, se obtiene $a=2$, $b=3$, $c=-1$, $d=-2$, de donde

$$A=\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$

$\therefore$ la ecuación homogénea asociada es

$$\dot{\overrightarrow{X}}_h=\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}\overrightarrow{X}_h$$

ii. Para encontrar el vector $\overrightarrow{B}$ observamos que $\overrightarrow{X}_p(t)=\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ es una solución particular a la ecuación no homogénea $\dot{\overrightarrow{X}}=A\overrightarrow{X}+\overrightarrow{B}$, esto es:

$$\dot{\overrightarrow{X}}_p=A\overrightarrow{X}_p+\overrightarrow{B}$$

$$\therefore\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix}+\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\therefore\ b_1 = -7$
$b_2 = 5$
$\therefore\ \overrightarrow{B} = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$

Ecuación:

$$\dot{\overrightarrow{X}} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \overrightarrow{X} + \begin{pmatrix} -7 \\ 5 \end{pmatrix}$$

4. (a) $\dot{\overrightarrow{X}} = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \overrightarrow{X}, \qquad \overrightarrow{X}(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$

$P_A(\lambda) = \lambda^2 + 4\lambda = \lambda(\lambda + 4) = 0$

$\therefore\ \lambda_1 = 0,\ \lambda_2 = -4$

$\lambda_1 = 0:$

$\therefore\ \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore\ -m + 3n = 0$

$\therefore\ m = 3n$

$n = 1 \Rightarrow m = 3 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda_2 = -4:$

$\begin{pmatrix} -1-(-4) & 3 \\ 1 & -3-(-4) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore\ \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore\ m + n = 0$

$\therefore\ n = -m$

$m = 1 \Rightarrow n = -1 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore\ \overrightarrow{X}(t) = k_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

Condición inicial:

$\overrightarrow{X}(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = k_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore\ 3k_1 + k_2 = 4$

$k_1 - k_2 = 0 \qquad \therefore \qquad k_1 = k_2 = 1$

Solución:

$\overrightarrow{X}(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

Límite:

Sabemos que

$$\lim_{t\to\infty} \; e^{-4t} = \lim_{t\to\infty} \; \frac{1}{e^{4t}} = 0.$$

Por lo tanto,

$$\lim_{t\to\infty}\overrightarrow{X}(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(b) $\dot{\overrightarrow{X}} = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}\overrightarrow{X}, \qquad \overrightarrow{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}$$

$$P_A(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0$$

$\therefore\; \lambda_1 = \lambda_2 = -2$

$\lambda_1 = -2:$

$$\begin{pmatrix} -3+2 & -1 \\ 1 & -1+2 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\; \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\; m + n = 0$

$\therefore\; n = -m$

$m = 1 \Rightarrow n = -1 \quad \therefore \quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{w}:$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore\; w_1 + w_2 = -1$

$\therefore\; w_2 = -w_1 - 1$

$w_1 = -1 \Rightarrow w_2 = 0 \quad \therefore \quad \vec{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\therefore\; \overrightarrow{X}(t) = k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-2t} + k_2\left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]e^{-2t}$$

Condición inicial:

$$\overrightarrow{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$\therefore\; k_1 - k_2 = 1$

$k_1 = 0$

$\therefore\; k_1 = 0,\;\; k_2 = -1$

Solución:

$$\overrightarrow{X}(t) = -\left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]e^{-2t}$$

Límite:

$$\lim_{t\to\infty}\overrightarrow{X}(t)=-\begin{pmatrix}1\\-1\end{pmatrix}\lim_{t\to\infty}te^{-2t}-\begin{pmatrix}-1\\0\end{pmatrix}\lim_{t\to\infty}e^{-2t}$$

Sabemos que

$$\lim_{t\to\infty}\ e^{-2t}=\lim_{t\to\infty}\ \frac{1}{e^{2t}}=0$$

$$\lim_{t\to\infty}\ te^{-2t}=\lim_{t\to\infty}\ \frac{t}{e^{2t}}\overset{\text{L'Hopital}}{=}\lim_{t\to\infty}\ \frac{1}{2e^{2t}}=0.$$

Por lo tanto,

$$\lim_{t\to\infty}\overrightarrow{X}(t)=\begin{pmatrix}0\\0\end{pmatrix}.$$

5. $\dot{\overrightarrow{X}}=\begin{pmatrix}0&1\\2&-1\end{pmatrix}\overrightarrow{X},\ \text{ con } \overrightarrow{X}(0)=\begin{pmatrix}3\\3w\end{pmatrix},\ w\in\Re$

$$A=\begin{pmatrix}0&1\\2&-1\end{pmatrix}$$

$$P_A(\lambda)=\lambda^2+\lambda-2=(\lambda+2)(\lambda-1)=0$$

$\therefore\ \lambda_1=-2,\ \lambda_2=1$

$\lambda_1=-2:$

$$\begin{pmatrix}0-(-2)&1\\2&-1-(-2)\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}2&1\\2&1\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\therefore\ 2m+n=0$

$\therefore\ n=-2m$

$m=1\Rightarrow n=-2\quad\therefore\quad \vec{v}_1=\begin{pmatrix}1\\-2\end{pmatrix}$

$\lambda_2=1:$

$$\begin{pmatrix}0-1&1\\2&-1-1\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}-1&1\\2&-2\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\therefore\ -m+n=0$

$\therefore\ n=m$

$m=1\Rightarrow n=1\quad\therefore\quad \vec{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$

$$\therefore\ \overrightarrow{X}(t)=k_1\begin{pmatrix}1\\-2\end{pmatrix}e^{-2t}+k_2\begin{pmatrix}1\\1\end{pmatrix}e^{t}$$

Condición inicial:

$$\overrightarrow{X}(0) = \begin{pmatrix} 3 \\ 3w \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore \; k_1 + k_2 = 3$

$-2k_1 + k_2 = 3w$

$\therefore \; k_1 = 1 - w, \;\; k_2 = 2 + w$

Solución:

$$\overrightarrow{X}(t) = (1-w) \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + (2+w) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

Límite:

Para que $\lim\limits_{t\to\infty} \overrightarrow{X}(t) = \overrightarrow{0}$ es necesario que $2 + w = 0$, es decir, $w = -2$. En ese caso,

$$\overrightarrow{X}(t) = 3 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$$

6. La solución del sistema $\dot{\overrightarrow{X}} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \overrightarrow{X} + \begin{pmatrix} -4 \\ 4 \end{pmatrix}$ es

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

El punto fijo $\overrightarrow{p}^*$ del sistema coincide con la soluciòn particular $\overrightarrow{X}_p = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, ya que ésta satisface $\dot{\overrightarrow{X}}_p = \overrightarrow{0}$. Así,

$$\overrightarrow{p}^* = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

Si se desea que $\lim\limits_{t\to\infty} \overrightarrow{X}(t) = \overrightarrow{p}^*$ es necesario que $k_1 = 0$. En ese caso,

$$\overrightarrow{X}(t) = k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\therefore \; \lim_{t\to\infty} \bar{X}(t) = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \overrightarrow{p}^*.$$

7. $\dot{\overrightarrow{X}} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \overrightarrow{X}$, con $\overrightarrow{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ y $\beta > 0$

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$P_A(\lambda) = \lambda^2 - 2\alpha\lambda + \left(\alpha^2 + \beta^2\right) = 0$$

$$\therefore\ \lambda_{1,2}=\frac{2\alpha\pm\sqrt{4\alpha^2-4\left(\alpha^2+\beta^2\right)}}{2}=\alpha\pm i\beta$$

$\lambda_1=\alpha+i\beta:$

$$\begin{pmatrix}\alpha-(\alpha+i\beta) & -\beta\\ \beta & \alpha-(\alpha+i\beta)\end{pmatrix}\begin{pmatrix}m\\ n\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}-i\beta & -\beta\\ \beta & -i\beta\end{pmatrix}\begin{pmatrix}m\\ n\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore\ -i\beta m-\beta n=0$$

$$\beta m-i\beta n=0$$

$$\therefore\ m=in$$

$$n=1\Rightarrow m=i\quad\therefore\ \vec{v}_1=\begin{pmatrix}i\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}+i\begin{pmatrix}1\\ 0\end{pmatrix}=\vec{r}+i\vec{s}$$

$$\therefore\ \vec{r}=\begin{pmatrix}0\\ 1\end{pmatrix},\ \vec{s}=\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$\therefore\ \overrightarrow{X}(t)=k_1e^{\alpha t}\left[\begin{pmatrix}0\\ 1\end{pmatrix}\cos(\beta t)-\begin{pmatrix}1\\ 0\end{pmatrix}sen\ (\beta t)\right]$$
$$+k_2e^{\alpha t}\left[\begin{pmatrix}0\\ 1\end{pmatrix}sen\ (\beta t)+\begin{pmatrix}1\\ 0\end{pmatrix}\cos(\beta t)\right]$$

Condición inicial:

$$\overrightarrow{X}(0)=\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}k_2\\ k_1\end{pmatrix}$$

$$\therefore\ k_1=0,\quad k_2=1$$

Solución:

$$\overrightarrow{X}(t)=e^{\alpha t}\begin{pmatrix}\cos(\beta t)\\ sen\ (\beta t)\end{pmatrix}$$

Convergencia:

La solución converge a largo plazo sólo si $\alpha<0$.

8. $\dot{\overrightarrow{X}}=\begin{pmatrix}c & 1\\ 0 & -1\end{pmatrix}\overrightarrow{X},\ \ c\in\mathbb{R},\ \ c\neq-1$

   (a) $A=\begin{pmatrix}c & 1\\ 0 & -1\end{pmatrix}$

   $$P_A(\lambda)=\lambda^2+(1-c)\lambda-c=0$$

   $$\therefore\ \lambda_{1,2}=\frac{-(1-c)\pm\sqrt{(1-c)^2+4c}}{2}=\frac{(c-1)\pm\sqrt{(c+1)^2}}{2}$$

$$\therefore\ \lambda_{1,2} = \frac{(c-1) \pm |c+1|}{2} = \frac{(c-1) \pm (c+1)}{2}$$

$\therefore\ \lambda_1 = c,\ \lambda_2 = -1$

(b) $\lambda_1 = c:$

$$\begin{pmatrix} c-c & 1 \\ 0 & -1-c \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 0 & 1 \\ 0 & -1-c \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ n = 0 \quad (m \text{ es libre})$

$$m = 1 \Rightarrow n = 0 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = -1:$

$$\begin{pmatrix} c-(-1) & 1 \\ 0 & -1-(-1) \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} c+1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ (c+1)\, m + n = 0$

$\therefore\ n = -(c+1)\, m$

$$m = 1 \Rightarrow n = -(c+1) \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -c-1 \end{pmatrix}$$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ct} + k_2 \begin{pmatrix} 1 \\ -c-1 \end{pmatrix} e^{-t}$$

(c) i. $-1 \neq c < 0:$

Sea $c = -|c|$. Por lo tanto,

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-|c|t} + k_2 \begin{pmatrix} 1 \\ -c-1 \end{pmatrix} e^{-t}$$

$$\therefore\ \lim_{t\to\infty} \overrightarrow{X}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

El sistema tiene un punto fijo en $\overrightarrow{p}^* = \overrightarrow{0}$.

ii. $c = 0:$

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

$$\therefore\ \lim_{t\to\infty} \overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

El sistema tiene una infinidad de puntos fijos estables a lo largo de la recta $k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $k_1 \in \mathbb{R}$.

iii. $c > 0$

Sea $c = |c|$. Por lo tanto,

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{|c|t} + k_2 \begin{pmatrix} 1 \\ -c-1 \end{pmatrix} e^{-t}$$

El sistema tiene un punto silla en $\overrightarrow{p}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

9. $\dot{w} = A(w - ap)$, $\dot{p} = B\dot{w} + C(w - ap)$, con $a, A, B, C > 0$ y $a(AB + C) > A$.

Sustituyendo la ecuación para $\dot{w}$ en la de $\dot{p}$ se obtiene:

$\dot{p} = B\dot{w} + C(w - ap) = B\left[A(w - ap)\right] + C(w - ap)$

$\therefore\ \dot{p} = (AB + C)\, w - a\,(AB + C)\, P$

Así, se llega al sistema

$\dot{w} = Aw - aAp$

$\dot{p} = (AB + C)\, w - a\,(AB + C)\, P,$

esto es,

$$\begin{pmatrix} \dot{w} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -aA \\ AB + C & -a\,(AB + C) \end{pmatrix} \begin{pmatrix} w \\ p \end{pmatrix}.$$

Puntos fijos ($\dot{w} = \dot{p} = 0$):

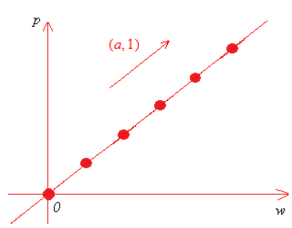
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A & -aA \\ AB + C & -a\,(AB + C) \end{pmatrix} \begin{pmatrix} w \\ p \end{pmatrix}$$

$\therefore\ Aw - aAp = 0$

$\therefore\ w^* = ap^*$

$\therefore\ \begin{pmatrix} w^* \\ p^* \end{pmatrix} = p^* \begin{pmatrix} a \\ 1 \end{pmatrix}$

Hay una infinidad de puntos fijos (M es singular).



Solución:

$$M = \begin{pmatrix} A & -aA \\ AB + C & -a\,(AB + C) \end{pmatrix}$$

$P_M(\lambda) = \lambda^2 - \lambda\left[A - a\,(AB + C)\right] = \lambda\left[\lambda - (A - a\,(AB + C))\right] = 0$

$\therefore\ \lambda_1 = 0,\ \lambda_2 = A - a\,(AB + C) < 0$

Se trata de un caso degenerado $(\lambda_1 = 0)$, estable $(\lambda_2 < 0)$.

$\lambda_1 = 0$:

$$\begin{pmatrix} A & -aA \\ AB+C & -a(AB+C) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \; Am - aAn = 0$

$\therefore \; m = an$

$$n = 1 \Rightarrow m = a \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} a \\ 1 \end{pmatrix}$$

$\lambda_2 = A - a(AB+C)$:

$$\begin{pmatrix} A - [A - a(AB+C)] & -aA \\ AB+C & -a(AB+C) - [A - a(AB+C)] \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a(AB+C) & -aA \\ AB+C & -A \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \; a(AB+C)m - aAn = 0$

$$\therefore \; n = \frac{AB+C}{A} m$$

$$m = A \Rightarrow n = AB + C \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} A \\ AB+C \end{pmatrix}$$

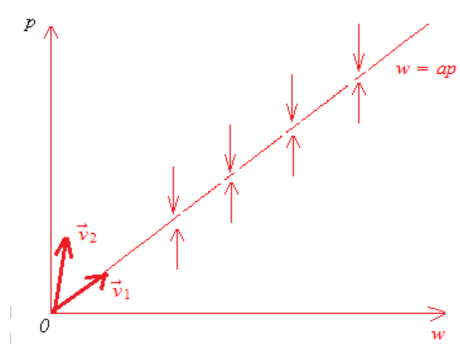
$$\therefore \begin{pmatrix} w(t) \\ p(t) \end{pmatrix} = k_1 \begin{pmatrix} a \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} A \\ AB+C \end{pmatrix} e^{\lambda_2 t}$$

Análisis del equilibrio:

Como $\lambda_2 < 0$, por lo tanto

$$\lim_{t\to\infty} \begin{pmatrix} w(t) \\ p(t) \end{pmatrix} = k_1 \begin{pmatrix} a \\ 1 \end{pmatrix}$$

El sistema tiene una infinidad de puntos fijos estables a lo largo de la recta $\begin{pmatrix} w \\ p \end{pmatrix} = k_1 \begin{pmatrix} a \\ 1 \end{pmatrix}$, $k_1 \in \mathbb{R}$.



Caso degenerado

10. $\dot{\pi} = \beta\pi + y, \quad \dot{y} = -\pi - \alpha, \quad \alpha, \beta \in \mathbb{R}$

Este es un sistema lineal no homogéneo, de la forma

$$\begin{pmatrix} \dot{\pi} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \pi \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha \end{pmatrix}.$$

$$A = \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix}$$

$$P_A(\lambda) = \lambda^2 - \beta\lambda + 1 = 0$$

$$\therefore \ \lambda_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}$$

(a) nodo atractor $\Longrightarrow \ \beta < 0 \ \cap \ \beta^2 - 4 > 0$

$\therefore \ \beta < 0 \ \cap \ |\beta| > 2$

$\therefore \ \beta < -2$

(b) punto silla $\Longrightarrow \det A < 0$

Como $\det A = 1 \geq 0$, por lo tanto no hay puntos silla.

11. $\dot{y} = ay - p, \quad \dot{p} = y - bp + ab - 1, \quad a, b > 0$

Este es un sistema lineal no homogéneo, de la forma

$$\begin{pmatrix} \dot{y} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & -b \end{pmatrix} \begin{pmatrix} y \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ ab - 1 \end{pmatrix}.$$

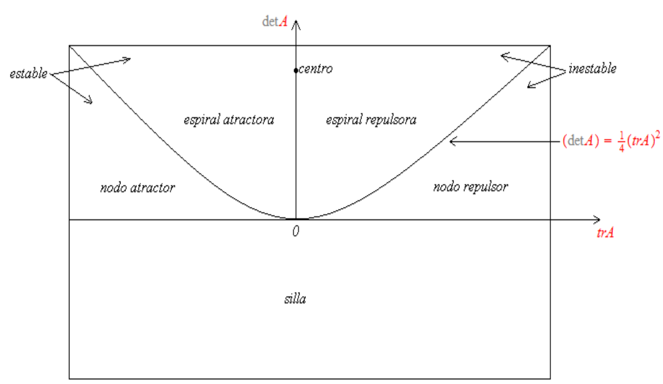
$$A = \begin{pmatrix} a & -1 \\ 1 & -b \end{pmatrix}$$

$$P_A(\lambda) = \lambda^2 - (trA)\lambda + (\det A) = 0$$

$$\therefore \ \lambda_{1,2} = \frac{(trA) \pm \sqrt{(trA)^2 - 4(\det A)}}{2}$$

$$trA = a - b$$

$$\det A = 1 - ab$$

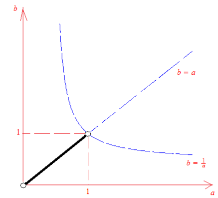


(a) cíclico(centro)$\Longrightarrow trA = 0$ y $\det A > 0$

$\therefore\ a - b = 0$ y $1 - ab > 0$

$\therefore\ b = a$ y $b < \dfrac{1}{a} \quad (a, b > 0)$

$\therefore\ b = a, \quad 0 < a < 1, \quad 0 < b < 1$

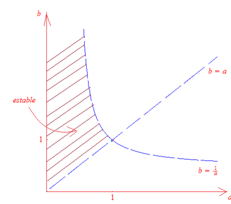


(b) estable(nodo o espiral atractores)$\Longrightarrow trA < 0$ y $\det A > 0$

$\therefore\ a - b < 0$ y $1 - ab > 0$

$\therefore\ a < b$ y $b < \dfrac{1}{a}$

$\therefore\ a < b < \dfrac{1}{a}, \quad 0 < a < 1$



12. (a) $\dot{\overrightarrow{X}} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \overrightarrow{X} + \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

i. Punto fijo$\left(\dot{\overrightarrow{X}} = \overrightarrow{0} \right)$:

Sea $\overrightarrow{p}^* = \begin{pmatrix} x^* \\ y^* \end{pmatrix}$.

$\therefore\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

$\therefore\ 0 = 2x^* - 4$

$0 = -2y^* + 6$

$\therefore\ x^* = 2, \ y^* = 3$

$\therefore\ \overrightarrow{p}^* = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Clasificación:

$A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$\therefore \det A = -4 < 0$

$\therefore$ se trata de un punto silla

ii. $\overrightarrow{X}(t) = \overrightarrow{X}_h(t) + \overrightarrow{X}_p(t)$

$\overrightarrow{X}_h$:

$$\dot{\overrightarrow{X}}_h = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \overrightarrow{X}_h$$

$$P_A(\lambda) = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2) = 0$$

$\therefore\ \lambda_1 = 2,\ \ \lambda_2 = -2$

$\lambda_1 = 2$:

$$\begin{pmatrix} 2-2 & 0 \\ 0 & -2-2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ n = 0$ (m es libre)

$$m = 1 \Rightarrow n = 0 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = -2$:

$$\begin{pmatrix} 2-(-2) & 0 \\ 0 & -2-(-2) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ m = 0$ (n es libre)

$$n = 1 \Rightarrow m = 0 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore\ \overrightarrow{X}_h(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$

$\overrightarrow{X}_p$:

$$\dot{\overrightarrow{X}}_p = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \overrightarrow{X}_p + \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

Proponemos $\overrightarrow{X}_p(t) = \begin{pmatrix} A \\ B \end{pmatrix} \quad \therefore \quad \dot{\overrightarrow{X}}_p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

$\therefore\ 0 = 2A - 4$

$0 = -2B + 6$

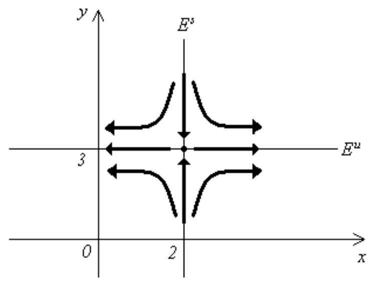
$\therefore\ A = 2, \quad B = 3$

$$\therefore\ \overrightarrow{X}_p(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \overrightarrow{p}^*$$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

iii.



$E^s\left(\overrightarrow{p}^*\right)=\{(x,y)\in\mathbb{R}^2|x=2\}$
$E^u\left(\overrightarrow{p}^*\right)=\{(x,y)\in\mathbb{R}^2|y=3\}$

(b) $\dot{\overrightarrow{X}}=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}\overrightarrow{X}+\begin{pmatrix}1\\-1\end{pmatrix}$

i. Punto fijo$\left(\dot{\overrightarrow{X}}=\overrightarrow{0}\right)$:

Sea $\overrightarrow{p}^*=\begin{pmatrix}x^*\\y^*\end{pmatrix}$.

$\therefore\ \begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}x^*\\y^*\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}$

$\therefore\ 0=-x^*+2y^*+1$
$0=y^*-1$

$\therefore\ x^*=3,\ \ y^*=1$

$\therefore\ \overrightarrow{p}^*=\begin{pmatrix}3\\1\end{pmatrix}$

Clasificación:

$A=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}$

$\therefore \det A=-1<0$

$\therefore$ se trata de un punto silla

ii. $\overrightarrow{X}(t)=\overrightarrow{X}_h(t)+\overrightarrow{X}_p(t)$

$\overrightarrow{X}_h$:

$\dot{\overrightarrow{X}}_h=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}\overrightarrow{X}_h$

$P_A(\lambda)=\lambda^2-1=(\lambda-1)(\lambda+1)=0$

$\therefore\ \lambda_1=1,\ \ \lambda_2=-1$

$\lambda_1=1$:

$\begin{pmatrix}-1-1 & 2\\ 0 & 1-1\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\therefore\ \begin{pmatrix}-2 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\therefore\ -2m+2n=0$

$\therefore\ n=m$

$$m = 1 \Rightarrow n = 1 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1$:

$$\begin{pmatrix} -1-(-1) & 2 \\ 0 & 1-(-1) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore\ n = 0$ (m es libre)

$$m = 1 \Rightarrow n = 0 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\therefore \overrightarrow{X}_h(t) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$$

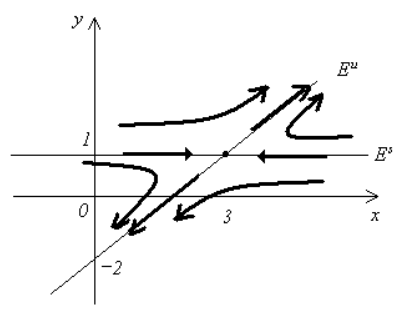
$\overrightarrow{X}_p$:

$$\overrightarrow{X}_p = \overrightarrow{p}^* = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

iii.



iv. $E^s:\ y - 1 = (0)(x-3)\ \therefore\ E^s(\overrightarrow{p}^*) = \{(x,y) \in \mathbb{R}^2 | y = 1\}$

$E^u:\ y - 1 = (1)(x-3)\ \therefore\ E^u(\overrightarrow{p}^*) = \{(x,y) \in \mathbb{R}^2 | y = x - 2\}$

(c) $\dot{\overrightarrow{X}} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \overrightarrow{X} - \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

i. Punto fijo$\left(\dot{\overrightarrow{X}} = \overrightarrow{0}\right)$:

Sea $\overrightarrow{p}^* = \begin{pmatrix} x^* \\ y^* \end{pmatrix}$.

$$\therefore \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix} - \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$\therefore\ 0 = y^*$

$0 = 2x^* + y^* - 6$

$\therefore\ x^* = 3,\ \ y^* = 0$

$$\therefore \overrightarrow{p}^* = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Clasificación:

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$\therefore \det A = -2 < 0$

$\therefore$ se trata de un punto silla

ii. $\overrightarrow{X}(t) = \overrightarrow{X}_h(t) + \overrightarrow{X}_p(t)$

$\overrightarrow{X}_h$:

$$\dot{\overrightarrow{X}}_h = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \overrightarrow{X}_h$$

$$P_A(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$\therefore \ \lambda_1 = 2, \ \ \lambda_2 = -1$

$\lambda_1 = 2$:

$$\begin{pmatrix} 0-2 & 1 \\ 2 & 1-2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \ \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \ -2m + n = 0$

$\therefore \ n = 2m$

$$m = 1 \Rightarrow n = 2 \quad \therefore \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\lambda_2 = -1$:

$$\begin{pmatrix} 0-(-1) & 1 \\ 2 & 1-(-1) \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \ \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \ m + n = 0$

$\therefore \ n = -m$

$$m = 1 \Rightarrow n = -1 \quad \therefore \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \ \overrightarrow{X}_h(t) = k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

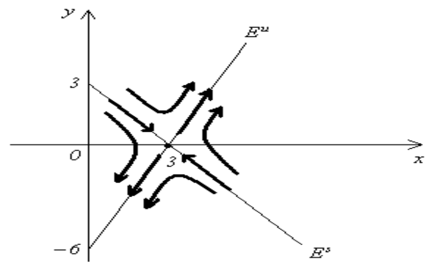
$\overrightarrow{X}_p$:

$$\overrightarrow{X}_p = \overrightarrow{p}^* = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Solución:

$$\overrightarrow{X}(t) = k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

iii.



iv. $E^s: \ y - 0 = (-1)(x-3) \ \therefore \ E^s(\overrightarrow{p}^*) = \{(x,y) \in \mathbb{R}^2 | y = 3 - x\}$
$E^u: \ y - 0 = (2)(x-3) \ \therefore \ E^u(\overrightarrow{p}^*) = \{(x,y) \in \mathbb{R}^2 | y = 2x - 6\}$

13. (a) $\dot{x} = -x + 2y = F(x,y)$

$\dot{y} = -2x - y = G(x,y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$-x + 2y = 0$

$-2x - y = 0$

$\therefore \ \overrightarrow{p}^* = (0,0)$

Clasificación:

$\lambda_{1,2} = -1 \pm 2i$

$\lambda_{1,2} \in \mathbb{C}$, con $\alpha < 0$

$\therefore$ Se trata de una espiral atractora

Isoclinas:

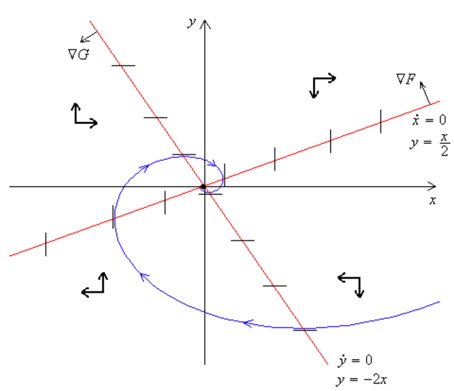
$\dot{x} = 0 \Rightarrow y = \dfrac{x}{2}$

$\dot{y} = 0 \Rightarrow y = -2x$

Gradientes:

$\nabla F = -\hat{\imath} + 2\hat{\jmath}$

$\nabla G = -2\hat{\imath} - \hat{\jmath}$



(b) $\dot{x} = -x + 5y = F(x, y)$

$\dot{y} = -x + y = G(x, y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$-x + 5y = 0$

$-x + y = 0$

$\therefore \ \overrightarrow{p}^* = (0, 0)$

Clasificación:

$\lambda_{1,2} = \pm 2i$

$\lambda_{1,2} \in \mathbb{C}$, con $\alpha = 0$

$\therefore$ Se trata de un caso degenerado (centro)

Isoclinas:

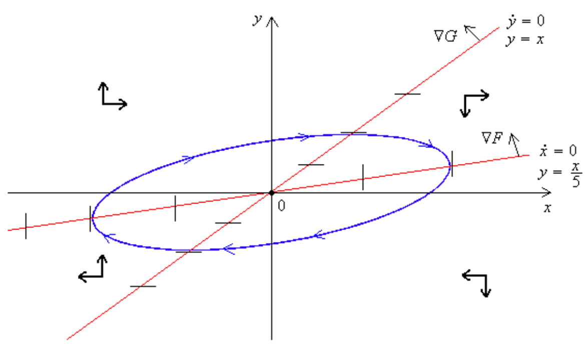
$\dot{x} = 0 \Rightarrow y = \dfrac{x}{5}$

$\dot{y} = 0 \Rightarrow y = x$

Gradientes:

$\nabla F = -\hat{\imath} + 5\hat{\jmath}$

$\nabla G = -\hat{\imath} + \hat{\jmath}$



(c) $\dot{x} = x + 3y - 4 = F(x, y)$

$\dot{y} = x - y + 4 = G(x, y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$x + 3y - 4 = 0$

$x - y + 4 = 0$

$\therefore \ \vec{p}^* = (-2, 2)$

Clasificación:

$\lambda_1 = 2,\ \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$; $\lambda_2 = -2,\ \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_1, \lambda_2 \in \mathbb{R}$, con $\lambda_1 > 0,\ \lambda_2 < 0$

$\therefore$ Se trata de un punto silla

Isoclinas:

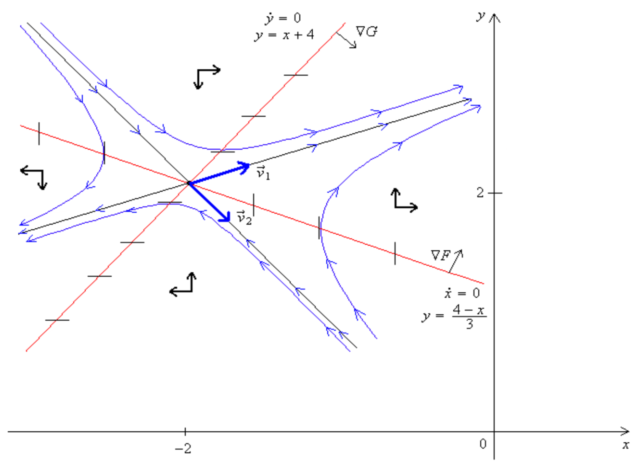
$\dot{x} = 0 \Rightarrow y = \dfrac{4-x}{3}$

$\dot{y} = 0 \Rightarrow y = x + 4$

Gradientes:

$\nabla F = \hat{\imath} + 3\hat{\jmath}$

$\nabla G = \hat{\imath} - \hat{\jmath}$



(d) $\dot{x} = 3x + y = F(x, y)$

$\dot{y} = x + 3y = G(x, y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$3x + y = 0$

$x + 3y = 0$

$\therefore \ \overrightarrow{p}^* = (0, 0)$

Clasificación:

$\lambda_1 = 2, \ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \ \lambda_2 = 4, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_1, \lambda_2 \in \mathbb{R}$, con $\lambda_1 \neq \lambda_2$, $\lambda_1 > 0$, $\lambda_2 > 0$

$\therefore$ Se trata de un nodo repulsor

Isoclinas:

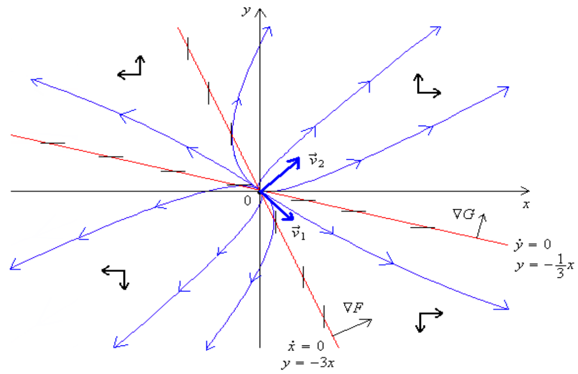
$\dot{x} = 0 \Rightarrow y = -3x$

$\dot{y} = 0 \Rightarrow y = -\dfrac{1}{3}x$

Gradientes:

$\nabla F = 3\hat{\imath} + \hat{\jmath}$

$\nabla G = \hat{\imath} + 3\hat{\jmath}$



(e) $\dot{x} = 4x - y = F(x, y)$

$\dot{y} = x + 2y = G(x, y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$4x - y = 0$

$x + 2y = 0$

$\therefore \ \overrightarrow{p}^* = (0, 0)$

Clasificación:

$\lambda_1 = \lambda_2 = 3, \ \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_1, \lambda_2 \in \mathbb{R}$, con $\lambda_1 = \lambda_2 > 0$

$\therefore$ Se trata de un caso degenerado repulsor

Isoclinas:

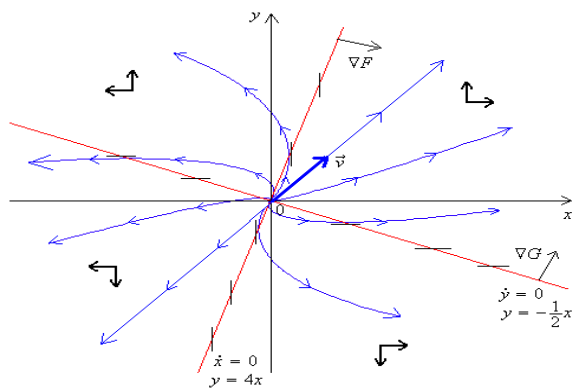
$\dot{x} = 0 \Rightarrow y = 4x$

$\dot{y} = 0 \Rightarrow y = -\dfrac{1}{2}x$

Gradientes:

$\nabla F = 4\hat{\imath} - \hat{\jmath}$

$\nabla G = \hat{\imath} + 2\hat{\jmath}$



(f) $\dot{x} = x + 3y = F(x,y)$

$\dot{y} = -2x - 6y = G(x,y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$x + 3y = 0$

$-2x - 6y = 0$

$\therefore\ \vec{p}^* = \left(x, -\dfrac{x}{3}\right) \longleftarrow$múltiples

Clasificación:

$\lambda_1 = 0,\ \vec{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix};\ \lambda_2 = -5,\ \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_1, \lambda_2 \in \mathbb{R}$, con $\lambda_1 = 0,\ \lambda_2 < 0$

$\therefore$ Se trata de un caso degenerado atractor

Isoclinas:

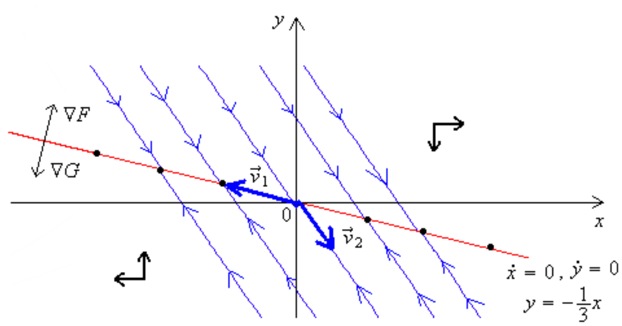
$\dot{x} = 0 \Rightarrow y = -\dfrac{1}{3}x$

$\dot{y} = 0 \Rightarrow y = -\dfrac{1}{3}x$

Gradientes:

$\nabla F = \hat{\imath} + 3\hat{\jmath}$

$\nabla G = -2\hat{\imath} - 6\hat{\jmath}$



14. (a) $\dot{x} = x - y = F(x,y)$

$\dot{y} = 1 - x^2 = G(x,y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$x - y = 0$

$1 - x^2 = 0$

$\therefore\ \vec{p}_1^* = (1,1),\ \ \vec{p}_2^* = (-1,-1)$

Clasificación:

$J(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2x & 0 \end{pmatrix}$

i. $J(1,1) = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

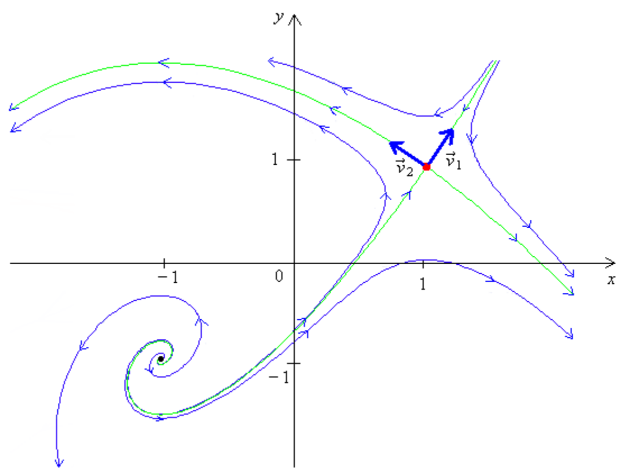
$\lambda_1 = -1,\ \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad \lambda_2 = 2,\ \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\therefore$ $(1,1)$ es un punto silla.

ii. $J(-1,-1) = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$

$\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2} i$

$\therefore$ $(-1,-1)$ es una espiral repulsora.



(b) $\dot{x} = 4x - 3xy = F(x,y)$

$\dot{y} = 3y - xy. = G(x,y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$x(4-3y) = 0$

$y(3-x) = 0$

$\therefore$ $\vec{p}_1^* = (0,0)$, $\vec{p}_2^* = \left(3, \frac{4}{3}\right)$

Clasificación:

$J(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 4-3y & -3x \\ -y & 3-x \end{pmatrix}$

i. $J(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

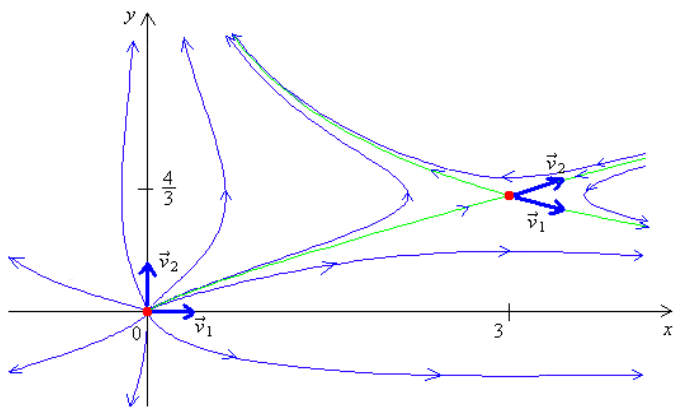
$\lambda_1 = 4,\ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \lambda_2 = 3,\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore$ $(0,0)$ es un nodo repulsor.

ii. $J\left(3, \frac{4}{3}\right) = \begin{pmatrix} 0 & -9 \\ -\frac{4}{3} & 0 \end{pmatrix}$

$$\lambda_1 = 2\sqrt{3}, \vec{v}_1 = \begin{pmatrix} 9 \\ -2\sqrt{3} \end{pmatrix}; \quad \lambda_2 = -2\sqrt{3}, \vec{v}_2 = \begin{pmatrix} 9 \\ 2\sqrt{3} \end{pmatrix}$$

$\therefore \left(3, \frac{4}{3}\right)$ es un punto silla.



15. (a) $\dot{x} = 2(y - e^x) = F(x, y)$

$\dot{y} = -x = G(x, y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$2(y - e^x) = 0$

$-x = 0$

$\therefore \vec{p}^* = (0, 1)$

Clasificación:

$$J(x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -2e^x & 2 \\ -1 & 0 \end{pmatrix}$$

$$J(0, 1) = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}$$

$P_J(\lambda) = \lambda^2 + 2\lambda + 2 = 0$

$\therefore \ \lambda_{1,2} = -1 \pm i$

$\therefore \ (0, 1)$ es una espiral atractora.

Isoclinas:

$\dot{x} = 0 \Rightarrow y = e^x$

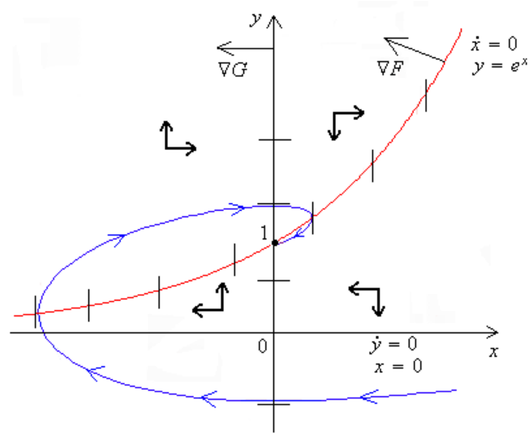
$\dot{y} = 0 \Rightarrow x = 0$

Gradientes:

$\nabla F(x, y) = -2e^x \ \hat{\imath} + 2\hat{\jmath}$

$\nabla G(x, y) = -\hat{\imath}$

Diagrama de fase:



(b) $\dot{x} = y - 1, \quad \dot{y} = 2(e^x - y)$.

$\dot{x} = y - 1 = F(x, y)$

$\dot{y} = 2(e^x - y) = G(x, y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$y - 1 = 0$

$2(e^x - y) = 0$

$\therefore \vec{p}^* = (0, 1)$

Isoclinas:

$\dot{x} = 0 \Rightarrow y = 1$

$\dot{y} = 0 \Rightarrow y = e^x$

Clasificación:

$$J(x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2e^x & -2 \end{pmatrix}$$

$$J(0, 1) = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}$$

$\det J = -2 < 0$

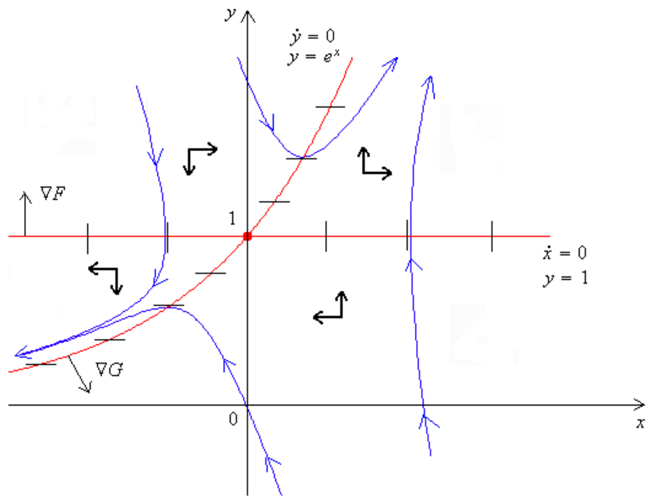
$\therefore (0, 1)$ es un punto silla.

Gradientes:

$\nabla F(x, y) = \hat{j}$

$\nabla G(x, y) = 2e^x \ \hat{\imath} - 2\hat{j}$

Diagrama de fase:



(c) $\dot{x} = y + \ln x, \qquad \dot{y} = 1 - x,$ con $x > 0$.

$\dot{x} = y + \ln x = F(x, y)$

$\dot{y} = 1 - x = G(x, y)$

Puntos fijos $(\dot{x} = \dot{y} = 0)$:

$y + \ln x = 0$

$1 - x = 0$

$\therefore \vec{p^*} = (1, 0)$

Clasificación:

$$J(x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} \frac{1}{x} & 1 \\ -1 & 0 \end{pmatrix}$$

$$J(1, 0) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$P_j(\lambda) = \lambda^2 - \lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$\therefore$ $(1, 0)$ es una espiral repulsora.

Isoclinas:

$\dot{x} = 0 \Rightarrow y = -\ln x$

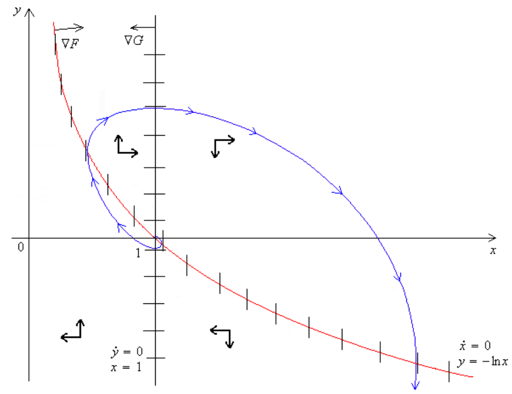
$\dot{y} = 0 \Rightarrow x = 1$

Gradientes:

$$\nabla F(x, y) = \frac{1}{x}\,\hat{\imath} + \hat{\jmath}$$

$$\nabla G(x, y) = -\hat{\imath}$$

Diagrama de fase:



16. (a) $\dot{x} = y - 2 = F(x,y)$

$\dot{y} = y - 2e^{-x} = G(x,y)$

i. Punto fijo ($\dot{x} = \dot{y} = 0$):

$y - 2 = 0$

$y - 2e^{-x} = 0$

$\therefore \vec{p}^* = (0,2)$

Clasificación:

$$J(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2e^{-x} & 1 \end{pmatrix}$$

$$J(0,2) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$\det J = -2 < 0$

$\therefore (0,2)$ es un punto silla.

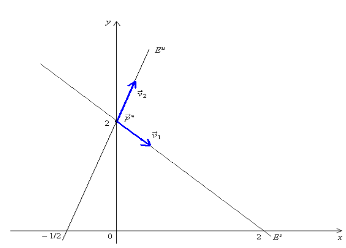
ii. Espacios:

$P_J(\lambda) = \lambda^2 - \lambda - 2 = 0$

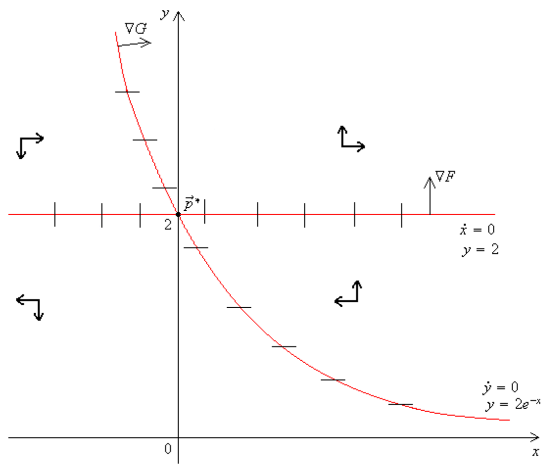
$$\therefore \ \lambda_1 = -1, \ \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ m^s = -1$$

$$\lambda_2 = 2, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ m^u = 2$$

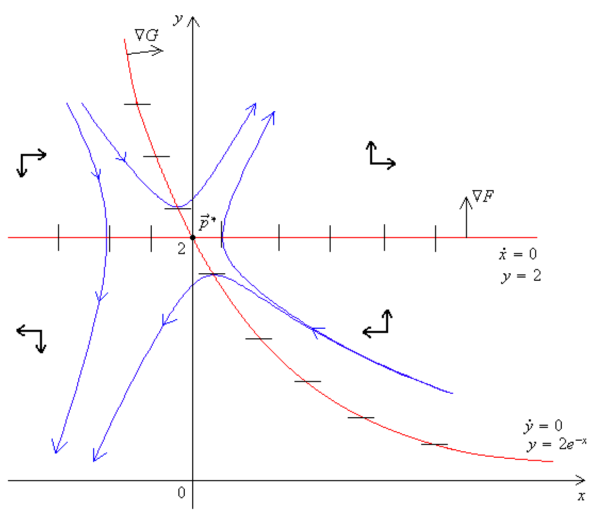
$$\therefore \begin{array}{l} E^s(0,2) = \{(x,y) \in \mathbb{R}^2 | y = -x + 2\} \\ E^u(0,2) = \{(x,y) \in \mathbb{R}^2 | y = 2x + 2\} \end{array}$$



iii. Isoclinas y flechitas:
$\dot{x} = 0 \Rightarrow y = 2$
$\dot{y} = 0 \Rightarrow y = 2e^{-x}$
Gradientes:
$\nabla F(x, y) = \hat{\jmath}$
$\nabla G(x, y) = 2e^{-x}\ \hat{\imath} + \hat{\jmath}$



iv. Diagrama de fase:



(b) $\dot{x} = 3 \ln x - 2y = F(x, y)$
$\dot{y} = 2(1 - x) = G(x, y)$

i. Punto fijo ($\dot{x} = \dot{y} = 0$):
$3 \ln x - 2y = 0$
$2(1 - x) = 0$
$\therefore \vec{p}^* = (1, 0)$

Clasificación:

$$J(x,y)=\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}=\begin{pmatrix} \dfrac{3}{x} & -2 \\ -2 & 0 \end{pmatrix}$$

$$J(1,0)=\begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix}$$

$\det J=-4<0$

$\therefore (1,0)$ es un punto silla.

ii. Espacios:

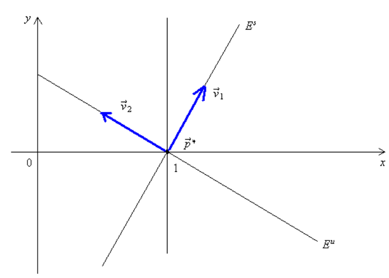
$P_J(\lambda)=\lambda^2-3\lambda-4=0$

$$\therefore\ \lambda_1=-1,\ \vec{v}_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \ m^s=2$$

$$\lambda_2=4,\ \vec{v}_2=\begin{pmatrix} -2 \\ 1 \end{pmatrix},\ \ m^u=-\frac{1}{2}$$

$$E^s(1,0)=\{(x,y)\in\mathbb{R}^2 | y=2x-2\}$$

$$\therefore\ E^u(1,0)=\left\{(x,y)\in\mathbb{R}^2 | y=-\frac{1}{2}x+\frac{1}{2}\right\}$$



iii. Isoclinas y flechitas:

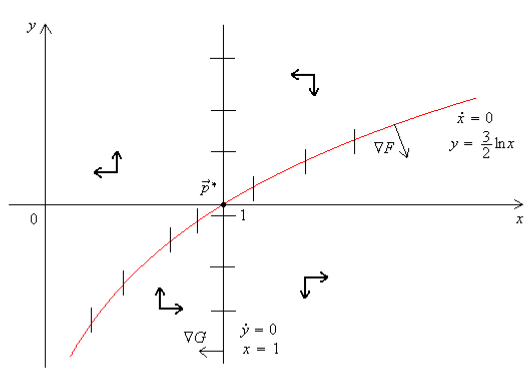
$\dot{x}=0\Rightarrow y=\dfrac{3}{2}\ln x$

$\dot{y}=0\Rightarrow x=1$

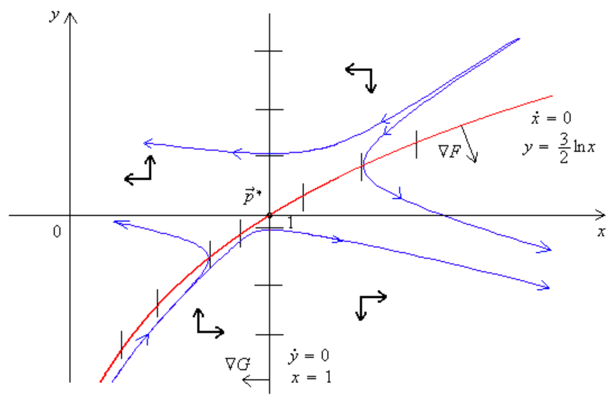
Gradientes:

$\nabla F(x,y)=\dfrac{3}{x}\ \hat{\imath}-2\hat{\jmath}$

$\nabla G(x,y)=-2\hat{\imath}$



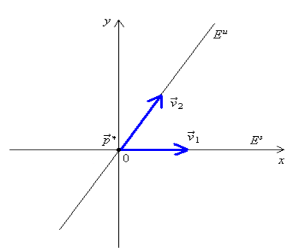
iv. Diagrama de fase:



(c) $\dot{x} = y + 1 - e^x = F(x, y)$
$\dot{y} = ye^x = G(x, y)$

i. Punto fijo ($\dot{x} = \dot{y} = 0$):
$y + 1 - e^x = 0$
$ye^x = 0$
$\therefore \vec{p}^* = (0, 0)$
Clasificación:
$$J(x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -e^x & 1 \\ ye^x & e^x \end{pmatrix}$$
$$J(0, 0) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$
$\det J = -1 < 0$
$\therefore (0, 0)$ es un punto silla.

ii. Espacios:
$P_J(\lambda) = \lambda^2 - 1 = 0$
$\therefore \ \lambda_1 = -1, \ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ m^s = 0$
$\lambda_2 = 1, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ m^u = 2$
$$\therefore \begin{array}{c} E^s(0, 0) = \{(x, y) \in \mathbb{R}^2 | y = 0\} \\ E^u(0, 0) = \{(x, y) \in \mathbb{R}^2 | y = 2x\} \end{array}$$



iii. Isoclinas y flechitas:

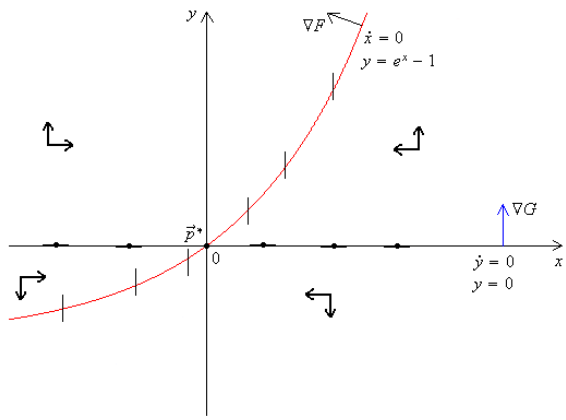
$\dot{x} = 0 \Rightarrow y = e^x - 1$

$\dot{y} = 0 \Rightarrow y = 0$

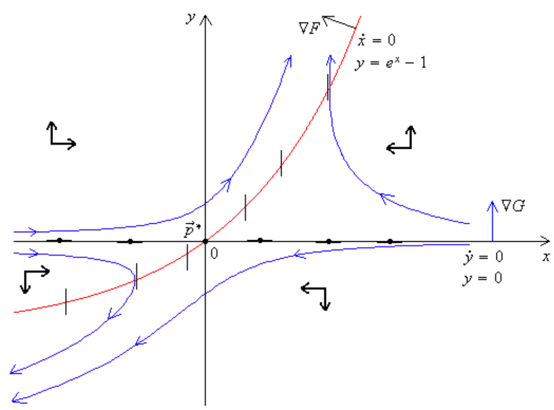
Gradientes:

$\nabla F(x, y) = -e^x \ \hat{\imath} + \hat{\jmath}$

$\nabla G(x, y) = ye^x \ \hat{\imath} + e^x \ \hat{\jmath}$



iv. Diagrama de fase:



(d) $\dot{x} = x - 1 = F(x, y)$

$\dot{y} = e^x - y = G(x, y)$

i. Punto fijo ($\dot{x} = \dot{y} = 0$):

$x - 1 = 0$

$e^x - y = 0$

$\therefore \vec{p}^* = (1, e)$

Clasificación:

$$J(x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^x & -1 \end{pmatrix}$$

$$J(1, e) = \begin{pmatrix} 1 & 0 \\ e & -1 \end{pmatrix}$$

$\det J = -1 < 0$

$\therefore (1, e)$ es un punto silla.

ii. Espacios:

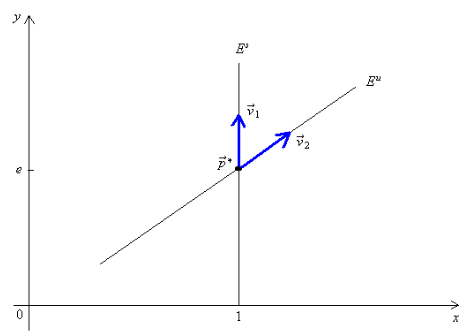
$P_J(\lambda) = \lambda^2 - 1 = 0$

$$\therefore \ \lambda_1 = -1, \ \ \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \ m^s \to \infty$$

$$\lambda_2 = 1, \ \ \vec{v}_2 = \begin{pmatrix} 2 \\ e \end{pmatrix}, \ \ m^u = \frac{e}{2}$$

$$E^s(1, e) = \{(x, y) \in \mathbb{R}^2 | x = 1\}$$

$$\therefore \ E^u(1, e) = \left\{(x, y) \in \mathbb{R}^2 | y = \frac{e}{2}x + \frac{e}{2}\right\}$$



iii. Isoclinas y flechitas:

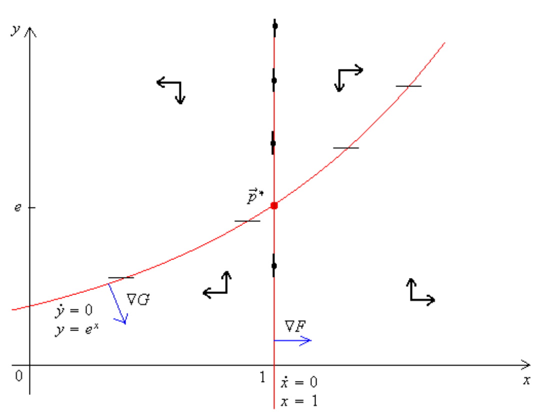
$\dot{x} = 0 \Rightarrow x = 1$

$\dot{y} = 0 \Rightarrow y = e^x$

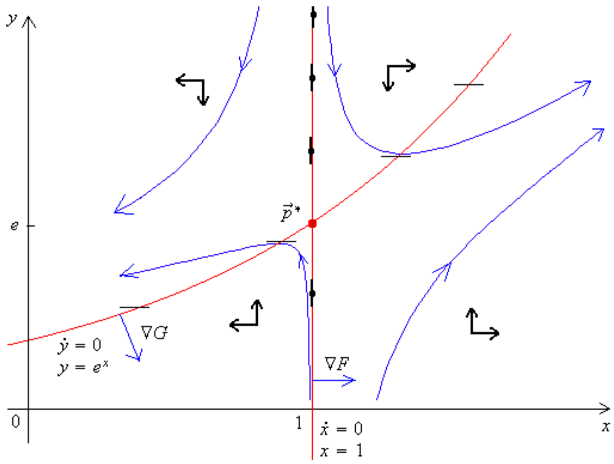
Gradientes:

$\nabla F(x, y) = \hat{\imath}$

$\nabla G(x, y) = e^x \ \hat{\imath} - \hat{\jmath}$



iv. Diagrama de fase:



(e) $\dot{x} = -2x + y = F(x, y)$

$\dot{y} = -\dfrac{6}{ax} + 3y = G(x, y), \qquad x, y > 0 \;\text{ y }\; a > 0$

i. Punto fijo ($\dot{x} = \dot{y} = 0$):

$-2x + y = 0$

$-\dfrac{6}{ax} + 3y = 0$

$\therefore y = 2x = \frac{2}{ax}$

$\therefore \vec{p}^* = \left(\dfrac{1}{\sqrt{a}}, \dfrac{2}{\sqrt{a}}\right)$

Clasificación:

$$J(x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \dfrac{6}{ax^2} & 3 \end{pmatrix}$$

$$J\left(\dfrac{1}{\sqrt{a}}, \dfrac{2}{\sqrt{a}}\right) = \begin{pmatrix} -2 & 1 \\ 6 & 3 \end{pmatrix}$$

$\det J = -12 < 0$

$\therefore \left(\dfrac{1}{\sqrt{a}}, \dfrac{2}{\sqrt{a}}\right)$ es un punto silla.

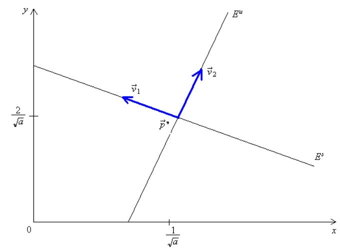
ii. Espacios:

$P_J(\lambda) = \lambda^2 - \lambda - 12 = 0$

$\therefore \;\lambda_1 = -3, \;\; \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \;\; m^s = -1$

$\lambda_2 = 4, \;\; \vec{v}_2 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}, \;\; m^u = 6$

$$\therefore \begin{array}{l} E^s\left(\frac{1}{\sqrt{a}},\frac{2}{\sqrt{a}}\right)=\left\{(x,y)\in\mathbb{R}^2|y=-x+\frac{3}{\sqrt{a}}\right\} \\ E^u\left(\frac{1}{\sqrt{a}},\frac{2}{\sqrt{a}}\right)=\left\{(x,y)\in\mathbb{R}^2|y=6x-\frac{4}{\sqrt{a}}\right\} \end{array}$$



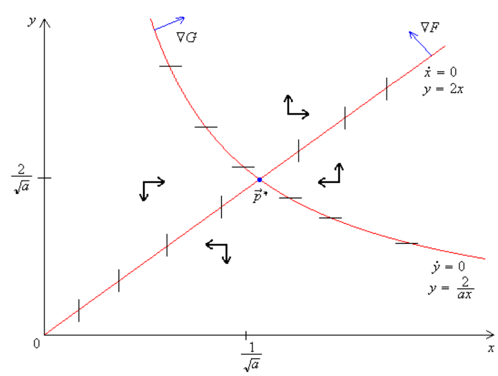
iii. $\dot{x}=0 \Rightarrow y=2x$

$\dot{y}=0 \Rightarrow y=\dfrac{2}{ax}$

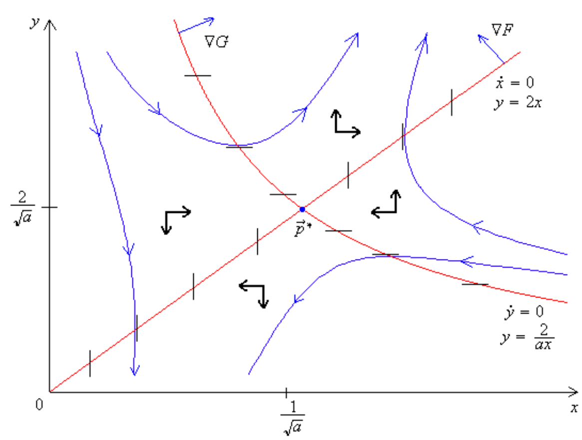
Gradientes:

$\nabla F(x,y)=-2\hat{\imath}+\hat{\jmath}$

$\nabla G(x,y)=\dfrac{6}{ax^2}\ \hat{\imath}+3\hat{\jmath}$



iv. Diagrama de fase:



17. $\dot{k} = f(k) - (n+\delta)k - c, \quad \dot{c} = \frac{c}{\theta}(f'(k) - \rho), \quad k(t) > 0,\, c(t) \;>\; 0,$ $n, \theta, \rho > 0$

Si $f(k) = k^{1/2}$, $n + \delta = 1/5$ y $\theta = \rho = 1/2$, entonces se tiene el sistema

$\dot{k} = k^{1/2} - \frac{1}{5}k - c = F(k, c)$

$\dot{c} = c(k^{-1/2} - 1) = G(k, c)$

(a) Puntos fijos $\left(\dot{k} = \dot{c} = 0\right)$:

$k^{1/2} - \frac{1}{5}k - c = 0$

$c(k^{-1/2} - 1) = 0$

$\therefore \; k = 1$

$\therefore \; c = 1 - \frac{1}{5} = \frac{4}{5}$

$\therefore (k^*, c^*) = \left(1, \frac{4}{5}\right)$

Clasificación:

$$J(k, c) = \begin{pmatrix} F_k & F_c \\ G_k & G_c \end{pmatrix} = \begin{pmatrix} \frac{1}{2k^{1/2}} - \frac{1}{5} & -1 \\ -\frac{c}{2k^{3/2}} & \frac{1}{k^{1/2}} - 1 \end{pmatrix}$$

$$J\left(1, \frac{4}{5}\right) = \begin{pmatrix} \frac{3}{10} & -1 \\ -\frac{2}{5} & 0 \end{pmatrix}$$

$\det J = -\frac{2}{5} < 0$

$\therefore \; \left(1, \frac{4}{5}\right)$ es un punto silla.

(b) $P_J(\lambda) = \lambda^2 - \frac{3}{10}\lambda - \frac{2}{5} = \left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{4}{5}\right) = 0$

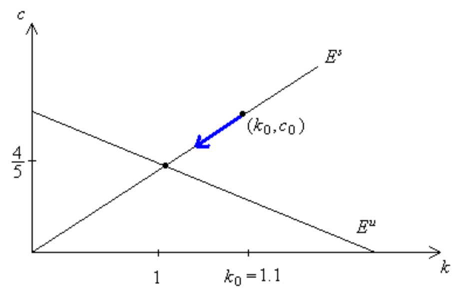
$\therefore \; \lambda_1 = -\frac{1}{2}, \, \vec{v}_1 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}; \; \lambda_2 = \frac{4}{5}, \, \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\therefore$ La dirección estable $(\lambda_1 < 0)$ es $\vec{v}_1 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ y la inestable $(\lambda_2 > 0)$ es $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

(c)
$$E^s\left(1, \frac{4}{5}\right) = \left\{(k, c) \in \mathbb{R}^2_{++} | c = \frac{4}{5}k\right\}$$
$$E^u\left(1, \frac{4}{5}\right) = \left\{(k, c) \in \mathbb{R}^2_{++} | c = -\frac{1}{2}k + \frac{13}{10}\right\}$$

Si $k_0 = 0.1 + k^* = 1.1$, para que el sistema converja al punto (k^*, c^*) tiene que seguir la trayectoria definida por el espacio estable $\left(c = \frac{4}{5}k\right)$, de modo que:

$$c_0 \cong \frac{4}{5}k_0 = \frac{4}{5}(1.1) = 0.88 > c^*$$



(d) $\dot{k} = k^{1/2} - \dfrac{1}{5}k - c = F(k, c)$

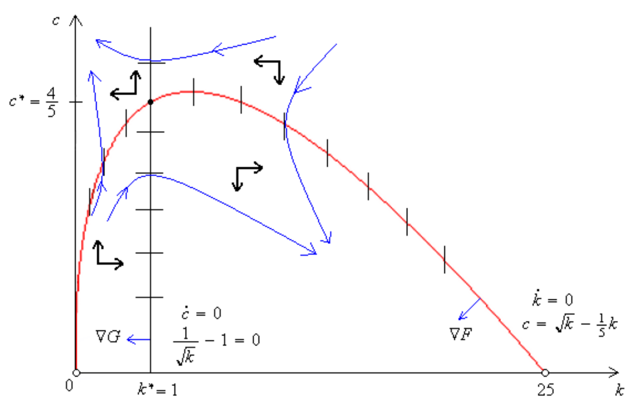
$\dot{c} = c(k^{-1/2} - 1) = G(k, c)$

Isoclinas:

$\dot{k} = 0 \Rightarrow c = \sqrt{k} - \frac{1}{5}k$

$\dot{c} = 0 \Rightarrow \dfrac{1}{\sqrt{k}} - 1 = 0$

$\therefore k = 1 = k^*$



Nota que el punto fijo en este ejemplo no es el máximo de la isoclina $c = \sqrt{k} - \dfrac{1}{5}k$.