Laboratorio 12

1. Calcular:

(a) $I = \int \frac{dx}{x^3+x^2-2x}$. Primero, observemos que $x^3+x^2-2x = x(x^2+x-2) = x(x-1)(x+2)$

$$\Rightarrow \frac{1}{x^3+x^2-2x} \underset{\exists A,B,C\in\mathbb{R}}{=} \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)} =$$

$$= \frac{Ax^2 + Ax - 2A + Bx^2 + 2Bx + Cx^2 - Cx}{x(x-1)(x+2)} \Rightarrow \begin{cases} A+B+C=0 \\ A+2B-C=0 \\ -2A=1 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow I = A\int \frac{dx}{x} + B\int\frac{dx}{x-1} + C\int\frac{dx}{x+2} = \log |x|^A|x-1|^B|x+2|^C + k,\ k\in\mathbb{R}$$

(b) $I = \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

Sea $f(x) = x^4-2x^2+4x+1$ y $g(x) = x^3-x^2-x+1$. Dado que $grad(f) > grad(g)$, tenemos una fracción propia.

$$\begin{array}{r|l} x^4 \quad -2x^2+4x+1 & x^3-x^2-x+1 \\ -x^4+x^3+x^2-x & \overline{-x-1} \\ \cancel{x^3}-\cancel{x^2}+3x+\cancel{1} & \\ -\cancel{x^3}+\cancel{x^2}+x-\cancel{1} & \\ 4x & \end{array}$$

En consecuencia tenemos que

$$\boxed{\frac{f(x)}{g(x)}} = \boxed{-x-1} + \boxed{\frac{4x}{x^3-x^2-x+1}}$$

Notemos que $x^3-x^2-x+1 = P_1(x)P_2(x)$ donde $P_{1,2}(x)$ son polinomios de grado 1 y 2, respectivamente. En este sentido, proponemos que $P_1(x) = x+1$

$$\Rightarrow P_2(x) = \frac{x^3-x^2-x+1}{x+1} \Rightarrow$$

$$\begin{array}{r|l} x+1 & \overline{-x^2+2x-1} \\ & \cancel{x^3}-x^2-x+1 \\ & -\cancel{x^3}-x^2 \\ & -\cancel{2x^2}-x+1 \\ & \cancel{2x^2}+2x \\ & \cancel{x}+\cancel{1} \\ & -\cancel{x}-\cancel{1} \\ & 0 \end{array}$$

En consecuencia, tenemos que

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x+1)(-x^2+2x-1)} = -\frac{4x}{(x+1)(x^2-2x+1)} = \boxed{\frac{4x}{(x+1)(x-1)^2}} \underset{\exists A,B,C\in\mathbb{R}}{=}$$

$$= \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + B(x^2-1) + C(x+1)}{(x+1)(x-1)^2} = \frac{(A+B)x^2 + (-2A+C)x + A - B + C}{(x+1)(x-1)^2}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

$$\Rightarrow \int \frac{4x}{(x+1)(x^2-1)}\,dx = A\log|x+1| + B\log|x-1| - \frac{C}{x-1} + k,\ k\in\mathbb{R}$$

⋮

(c) $I = \int \frac{3x^3-3x^2+5x+3}{1-x^4}\,dx$

Sea $f(x) = 3x^3-3x^2+5x+3$ y $g(x) = 1-x^4$. Tenemos que $\frac{f(x)}{g(x)}$ es fracción impropia.

Además, notemos que $g(x) = (1-x^2)(1+x^2) = (1-x)(1+x)(1+x^2)$, entonces existen $A,B,C,D\in\mathbb{R}$ tales que

$$\frac{1}{g(x)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2} \dots \quad \int \frac{f(x)}{g(x)}dx = A\int\frac{f(x)}{1-x}dx + B\int\frac{f(x)}{1+x}dx + \int\frac{(Cx+D)f(x)}{1+x^2}dx$$

⋮

2. Determinar $\int \frac{dx}{(x^2+1)^2}$ y calcular $\int \frac{2-x+x^2-x^3}{(x^2+1)^2}dx$

$$\int\frac{dx}{(x^2+1)^2} = \int\frac{\sec^2\theta}{(\tan^2\theta+1)^2}d\theta = \int\frac{\cancel{\sec^2\theta}}{\sec^{\cancel{4}2}\theta}d\theta = \int\cos^2\theta\, d\theta = \dots$$

$\begin{cases} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{cases}$; $\tan^2\theta + 1 = \sec^2\theta$

Observemos, en consecuencia, que

$$\int\frac{2-x+x^2-x^3}{(x^2+1)^2}dx = \int(2-\tan\theta+\tan^2\theta-\tan^3\theta)\cos^2\theta\, d\theta = \int 2\cos^2\theta\, d\theta - \int \operatorname{sen}\theta\cos\theta\, d\theta +$$

$\begin{cases} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{cases}$

$$+\int\operatorname{sen}^2\theta\, d\theta - \int\frac{\operatorname{sen}^3\theta}{\cos\theta}d\theta$$

$$= \int -\frac{(1-u^2)}{u}du$$

$\begin{cases} u = \cos\theta \\ du = -\operatorname{sen}\theta\, d\theta \end{cases}$; $\operatorname{sen}\theta = \sqrt{1-u^2}$

3. Determinar las primitivas siguientes

(a) $$\int\frac{dx}{\operatorname{senh}x} = \int \frac{\pm 1}{\sqrt{u^2-1}}\left(\pm\frac{1}{\sqrt{u^2-1}}\right)du = \int\frac{du}{u^2-1} \underset{\exists A,B\in\mathbb{R}}{=} \int\frac{A}{u-1}du + \int\frac{B}{u+1}du \dots$$

$\begin{cases} u = \cosh x \\ du = \operatorname{senh}x\, dx \Rightarrow du = \pm\sqrt{u^2-1}\,dx \end{cases}$; $\cosh^2 x - \operatorname{senh}^2 x = 1 \Rightarrow \operatorname{senh}x = \pm\sqrt{u^2-1}$

(b) $$\int\frac{dx}{\operatorname{sen}x(1+\cos x)} = \int \mp\frac{1}{\sqrt{1-u^2}}\,\frac{1}{\pm\sqrt{1-u^2}(1+u)}du = \int\frac{du}{(1-u^2)(1+u)} = \int\frac{du}{(1-u)(1+u)^2} =$$

$\begin{cases} u = \cos x \\ du = -\operatorname{sen}x\, dx \end{cases}$; $\operatorname{sen}^2 x + \cos^2 x = 1 \Rightarrow \operatorname{sen}x = \pm\sqrt{1-u^2}$ $\Rightarrow dx = \mp\frac{1}{\sqrt{1-u^2}}du$

$1-u^2 = (1-u)(1+u)$

$$\underset{\exists A,B,C\in\mathbb{R}}{=} \int\frac{A}{1-u}du + \boxed{\int\frac{B}{1+u}du + \int\frac{C}{(1+u)^2}du} = -A\log|1-u| + B\log|1+u| - \frac{C}{1+u} + k,$$ $k\in\mathbb{R}$.

$$\Rightarrow \int\frac{dx}{\operatorname{sen}x(1+\cos x)} = \log|1+\cos x|^B - \log|1-\cos x|^A - \frac{C}{1+\cos x} + k,\ k\in\mathbb{R}.$$

4. Calcular $\lim_{\beta\to\infty}\int_1^\beta\frac{dx}{ax(bx+c)}$, donde $a,b,c>0$.

Observemos que
$$\int_1^\beta\frac{dx}{ax(bx+c)} = \int_{b/c}^{b/c\,\beta}\frac{\cancel{c/b}\,du}{a\,\cancel{\frac{c}{b}}u(u+1)c} = \frac{1}{ac}\int_{b/c}^{b/c\,\beta}\frac{du}{u(u+1)} = \frac{1}{ac}\log\left|\frac{\cancel{b/c}\,\beta}{\cancel{b/c}}\right| - \frac{1}{ac}\log\left|\frac{b/c\,\beta+1}{b/c+1}\right|$$

$\begin{cases} bx = cu \\ dx = \frac{c}{b}du \end{cases}$; $u(1) = \frac{b}{c}$, $u(\beta) = \frac{b}{c}\beta$; $\frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1}$

$$= \frac{1}{ac}\log\left|\frac{\beta\frac{b}{c}+\beta}{b/c\,\beta+1}\right| = \frac{1}{ac}\log\left|\frac{\frac{b}{c}+1}{b/c+1/\beta}\right| \xrightarrow{\beta\to\infty} \frac{1}{ac}\log\left(1+\frac{c}{b}\right)$$

5. Calcular $L = \lim_{\beta\to\infty}\int_1^\beta\frac{dx}{e^{2x}-e^x}$.

Observemos que $\frac{1}{e^{2x}-e^x} = \frac{1}{e^x(e^x-1)} \underset{u=e^x}{=} \frac{1}{u(u-1)}$. De este modo, notemos que

$$\int_1^\beta\frac{dx}{e^x(e^x-1)} = \int_e^{e^\beta}\frac{du}{u^2(u-1)} = \int_e^{e^\beta}\frac{A}{u}du + \int_e^{e^\beta}\frac{B}{u^2}du + \int_e^{e^\beta}\frac{C}{u-1}du = A\log\frac{e^\beta}{e} - B\left(\frac{1}{e^\beta}-\frac{1}{e}\right) + C\log\left|\frac{e^\beta-1}{e-1}\right| =$$

$du = e^x dx \Rightarrow dx = \frac{du}{u}$; $\exists A,B,C\in\mathbb{R}$

$$= \log\left(\frac{e^\beta}{e}\right)^A\left|\frac{e^\beta-1}{e-1}\right|^C + B\left(\frac{1}{e}-\frac{1}{e^\beta}\right)$$

Los valores de A, B y C están dados a partir del siguiente cálculo

$$\frac{1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1} = \frac{Au(u-1) + B(u-1) + Cu^2}{u^2(u-1)} = \frac{(A+C)u^2 + (-A+B)u - B}{u^2(u-1)}$$

$$\Rightarrow \begin{cases} B = -1 \\ A = -1 \\ C = 1 \end{cases} \quad \therefore\ L = \lim_{\beta\to\infty}\left(\log\left|\frac{e}{e-1}\,\frac{e^\beta-1}{e^\beta}\right| + \frac{1}{e^\beta} - \frac{1}{e}\right) = \log\frac{e}{e-1} - \frac{1}{e}$$