Laboratorio 13

1. Calcular el valor de las integrales siguientes o mostrar divergencia.

(a) $I = \int_1^\infty \frac{dx}{x(1+5x)}$

Observemos que la función $f(x) = \frac{1}{x(1+5x)}$ es continua en $x \geq 1$. Por otro lado, notemos que $1+5x>0 \ \forall x \geq 1 \Rightarrow x(1+5x) = x+5x^2 > 5x^2 \ \forall x \geq 1$.
En consecuencia, tenemos que $f(x) = \frac{1}{x(1+5x)} < \frac{1}{5x^2}$. Por la propiedad de homogeneidad de la integral, tenemos que

$$0 < I = \int_1^\infty \frac{dx}{x(1+5x)} = \lim_{\alpha\to\infty}\int_1^\alpha \frac{dx}{x(1+5x)} < \lim_{\alpha\to\infty}\int_1^\alpha \frac{dx}{5x^2} = \lim_{\alpha\to\infty}\frac{1}{5}\left(1-\frac{1}{\alpha}\right) = \frac{1}{5}$$

Es decir, I existe, entonces notemos que existen $A, B \in \mathbb{R}$ tales que

$$f(x) = \frac{A}{x} + \frac{B}{1+5x} = \frac{A+(5A+B)x}{x(1+5x)} \Rightarrow A=1 \text{ y } B=-5A=-5.$$

De este modo, obtenemos que

$$I = \int_1^\infty \left(\frac{1}{x} - \frac{5}{1+5x}\right)dx = \lim_{\alpha\to\infty}\int_1^\alpha \left(\frac{1}{x}-\frac{5}{1+5x}\right)dx = \lim_{\alpha\to\infty}\left[\log x\Big|_1^\alpha - \log(1+5x)\Big|_1^\alpha\right] =$$

$$= \lim_{\alpha\to\infty}\log\frac{6\alpha}{1+5\alpha} = \log\frac{6}{5}$$

(b) $I = \int_{\log 2}^\infty \frac{e^{-x}}{1-e^{-2x}}dx$

Observemos que $\lim_{x\to\infty}\frac{e^{-x}}{1-e^{-2x}} = 0$, esto indica que $f(x)$ no sólo es continua en $x \geq \log 2$, sino que la integral I es plausible de existir

$$I = \int_{\log 2}^\infty \frac{e^{-x}}{1-e^{-2x}}dx = \lim_{\alpha\to\infty}\int_{\log 2}^\alpha \frac{e^{-x}}{1-e^{-2x}}dx = \lim_{\alpha\to\infty}\int_{1/2}^{e^{-\alpha}}\frac{du}{u^2-1}$$

$$\begin{cases} u = e^{-x} \\ du = -e^{-x}dx = -u\,dx \end{cases} \Rightarrow \begin{matrix} u(\log 2) = 1/2 \\ u(\alpha) = e^{-\alpha} \end{matrix}$$

Observemos que $\frac{1}{u^2-1} = \frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}$

$$\Rightarrow \int_{1/2}^{e^{-\alpha}}\frac{du}{u^2-1} = \frac{1}{2}\log\frac{e^{-\alpha}-1}{1/2-1} - \frac{1}{2}\log\frac{e^{-\alpha}+1}{1/2+1} = \frac{1}{2}\log\left(\frac{1-e^{-\alpha}}{1+e^{-\alpha}}\frac{1}{3}\right) \xrightarrow{\alpha\to\infty} \frac{1}{2}\log\frac{1}{3}$$

$$\therefore I = -\frac{1}{2}\log 3$$

(c) $I = \int_0^\infty \frac{dx}{e^x+e^{-x}}$

La función $f(x) = \frac{1}{e^x+e^{-x}}$ es continua en $x \geq 0$ y $\lim_{x\to\infty} f(x) = 0$. De este modo, la integral I plausiblemente converge, entonces

$$I = \lim_{\alpha\to\infty}\int_0^\alpha \frac{dx}{e^x+e^{-x}} = \lim_{\alpha\to\infty}\int_1^{e^\alpha}\frac{du}{u^2+1} = \lim_{\alpha\to\infty}\left[\arctan e^\alpha - \arctan 1\right] = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$$

$$\begin{cases} u = e^x \\ du = e^x dx \end{cases} \Rightarrow \begin{matrix} u(0) = 1 \\ u(\alpha) = e^\alpha \end{matrix}$$

(d) $I = \int_{-\infty}^0 x e^{2x}dx$ Observemos que $f(x) = xe^{2x}$ es continua en $x \leq 0$ y tiene una asíntota horizontal nula puesto que $\lim_{x\to-\infty} f(x) = 0$.

$$I = \int_0^\infty -ue^{-2u}du = -\left(-\frac{1}{2}e^{-2u}\right)u\Big|_0^\infty - \int_0^\infty \frac{1}{2}e^{-2u}du = \frac{1}{4}e^{-2u}\Big|_0^\infty = -\frac{1}{4}$$

$\begin{cases} u = -x \\ du = -dx \end{cases}$ por partes

(e) $I = \int_{-\infty}^\infty |x|e^{-x^2}dx$. La función $f(x) = |x|e^{-x^2}$ es continua en $\mathbb{R}$, además $f(-x) = f(x)$, lo cual indica que $f(x)$ es par; por otro lado $\lim_{x\to\pm\infty} f(x) = 0$.

$$I = \lim_{\alpha\to\infty}\int_{-\alpha}^\alpha |x|e^{-x^2}dx = 2\lim_{\alpha\to\infty}\int_0^\alpha xe^{-x^2}dx = -\lim_{\alpha\to\infty}\int_0^\alpha -2xe^{-x^2}dx = \lim_{\alpha\to\infty}(1-e^{-2\alpha}) = 1$$

(f) $I = \int_0^1 x\log x\,dx = \frac{1}{2}x^2\log x\Big|_0^1 - \frac{1}{2}\int_0^1 x\,dx = \lim_{\alpha\to 0^+} -\frac{1}{2}\alpha^2\log\alpha - \frac{1}{4}$

$$\begin{cases} f(x) = \log x \\ g'(x) = x \end{cases} \Rightarrow \begin{matrix} f'(x) = \frac{1}{x} \\ g(x) = \frac{1}{2}x^2 \end{matrix}$$

De este modo, tenemos que

$$\lim_{\alpha\to 0^+} -\frac{1}{2}\alpha^2\log\alpha = -\frac{1}{2}\lim_{\alpha\to 0^+}\frac{\log\alpha}{1/\alpha^2} \underset{L'H}{=} -\frac{1}{2}\lim_{\alpha\to 0^+}\frac{1/\alpha}{-\frac{2}{\alpha^3}} = \frac{1}{4}\lim_{\alpha\to 0^+}\alpha^2 = 0$$

$$\therefore I = -\frac{1}{4}$$

(l) $I = \int_a^b \frac{dx}{\sqrt{x-a}\sqrt{b-x}}$, $a<b$.

Observemos que $f(x) = \frac{1}{\sqrt{(x-a)(b-x)}}$ es continua en (a,b) y $\lim_{\substack{x\to a^+ \\ x\to b^-}} f(x)$ diverge, entonces I es impropia.

Sea $x-a = (b-a)\operatorname{sen}^2\theta \Rightarrow b-x = b-a-(b-a)\operatorname{sen}^2\theta = (b-a)(1-\operatorname{sen}^2\theta) = (b-a)\cos^2\theta$

$$\Rightarrow (x-a)(b-x) = (b-a)^2\operatorname{sen}^2\theta\cos^2\theta = \frac{(b-a)^2}{4}\operatorname{sen}^2 2\theta$$

$$\Rightarrow \frac{1}{\sqrt{(x-a)(b-x)}} = \frac{2}{b-a}\frac{1}{|\operatorname{sen}2\theta|}.$$

Por otro lado, $dx = 2(b-a)\operatorname{sen}\theta\cos\theta\,d\theta = (b-a)\operatorname{sen}2\theta\,d\theta$ y, además, tenemos que: si $x=a$, entonces $0 = a-a = (b-a)\operatorname{sen}^2\theta \Rightarrow \operatorname{sen}\theta = 0 \Rightarrow \theta = 0$
si $x=b$, entonces $b-a = (b-a)\operatorname{sen}^2\theta \Rightarrow \operatorname{sen}^2\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

De esta manera, tenemos que $I = \int_0^{\pi/2} 2\frac{\cancel{(b-a)}\operatorname{sen}2\theta\,d\theta}{\cancel{(b-a)}|\operatorname{sen}2\theta|} = 2\int_0^{\pi/2}d\theta = \pi$

2. Probar que $\int_0^1(\log x)^n dx = (-1)^n n!$, donde $n! = n(n-1)\cdots 3\cdot 2\cdot 1$.
$(-1)(-1)(-1)\cdots = (-1)^n$

Observemos que $\log x$ es continuo en $(0,1]$, entonces $\int_0^1(\log x)^n dx = \lim_{\alpha\to 0^+}\int_\alpha^1(\log x)^n dx$

Tenemos que $I = \int_\alpha^1(\log x)^n dx = \boxed{\int_{\log\alpha}^0 u^n e^u du} = u^n e^u\Big|_{\log\alpha}^0 - n\boxed{\int_{\log\alpha}^0 u^{n-1}e^u du} = -(\log\alpha)^n\alpha - n\int_{\log\alpha}^0 u^{n-1}e^u du$

$$\begin{cases} u = \log x \\ du = \frac{dx}{x} \Rightarrow dx = e^u du \end{cases} \Rightarrow \begin{matrix} u(\alpha) = \log\alpha \\ u(1) = 0 \end{matrix} \qquad \begin{cases} f'(u) = e^u \\ g(u) = u^n \end{cases} \Rightarrow \begin{matrix} f(u) = e^u \\ g'(u) = nu^{n-1} \end{matrix}$$

Observemos que $\lim_{\alpha\to 0^+}\alpha(\log\alpha)^n = \lim_{\alpha\to 0^+}\frac{(\log\alpha)^n}{1/\alpha} \underset{L'H}{=} \lim_{\alpha\to 0^+}\frac{\frac{n}{\alpha}(\log\alpha)^{n-1}}{-1/\alpha^2} = \lim_{\alpha\to 0^+} -n\frac{(\log\alpha)^{n-1}}{1/\alpha} = \cdots = 0.$ L'H $(n-1)$-veces

3. Encontrar el valor de $A \in \mathbb{R}$ tal que determine el valor de

$$I = \int_2^\infty\left(\frac{x}{2x^2+1} - \frac{A}{x+1}\right)dx = \lim_{\alpha\to\infty}\int_2^\alpha\left(\frac{4x}{4(2x^2+1)} - \frac{A}{x+1}\right)dx = \lim_{\alpha\to\infty}\left[\frac{1}{4}\log(2x^2+1)\Big|_2^\alpha - A\log(x+1)\Big|_2^\alpha\right]$$

$$= \lim_{\alpha\to\infty}\left[\frac{1}{4}\log\left(\frac{2\alpha^2+1}{9}\right) - A\log\frac{\alpha+1}{3}\right] = \lim_{\alpha\to\infty}\log\left[\left(\frac{2\alpha^2+1}{9}\right)^{1/4}\left(\frac{3}{\alpha+1}\right)^A\right] =$$

$$= \lim_{\alpha\to\infty}\left[\log\frac{3^A}{9^{1/4}} + \log\frac{(2\alpha^2+1)^{1/4}}{(\alpha+1)^A}\right]$$

Sea $A = \frac{1}{2}$, entonces $\lim_{\alpha\to\infty}\log\frac{(2\alpha^2+1)^{1/4}}{\sqrt{\alpha+1}} = \log\left(\lim_{\alpha\to\infty}\left(\frac{2\alpha^2+1}{(\alpha+1)^2}\right)^{1/4}\right)$. Ahora, observemos
$\log(\cdot)$ continua

que $\lim_{\alpha\to\infty}\frac{2\alpha^2+1}{\alpha^2+2\alpha+1} = 2 \Rightarrow \lim_{\alpha\to\infty}\log\frac{(2\alpha^2+1)^{1/4}}{\sqrt{\alpha+1}} = \log\sqrt[4]{2} = \frac{1}{4}\log 2$

$$\therefore I = \lim_{\alpha\to\infty}\left[\log\frac{3^{1/2}}{9^{1/4}} + \frac{1}{4}\log 2\right] = \frac{1}{4}\log 2$$

1. (g) $I = \int_0^{\pi/2}\frac{dx}{1-\operatorname{sen}x} = \int_0^{\pi/2}\frac{1+\operatorname{sen}x}{1-\operatorname{sen}^2x}dx = \int_0^{\pi/2}(\sec^2x + \sec x\tan x)dx = \lim_{\alpha\to\frac{\pi}{2}^-}(\tan\alpha + \sec\alpha) = \cdots$
$1-\operatorname{sen}^2x = \cos^2x$

(h) $I = \int_0^1\frac{e^x}{e^x-1}dx = \lim_{\alpha\to 0^+}\log\frac{e-1}{e^\alpha-1}\ \cdots$

(i) $I = \int_0^1\frac{4r}{\sqrt{1-r^4}}dr = \int_0^1\frac{2\,du}{\sqrt{1-u^2}} = \int_0^{\pi/2}\frac{2\cos\theta\,d\theta}{\sqrt{1-\operatorname{sen}^2\theta}}\ \cdots$

$$\begin{cases} u = r^2 \\ du = 2r\,dr \end{cases} \Rightarrow \begin{matrix} u(0) = 0 \\ u(1) = 1 \end{matrix} \qquad \begin{cases} u = \operatorname{sen}\theta \\ du = \cos\theta\,d\theta \end{cases} \Rightarrow \begin{matrix} 0 = \operatorname{sen}(0) \\ 1 = \operatorname{sen}(\pi/2) \end{matrix}$$

(j) $\int_1^{\cosh t}\frac{dx}{\sqrt{x^2-1}}$, $t \geq 0$.

Observemos que, si $x = \cosh\theta$, entonces $dx = \operatorname{senh}\theta\,d\theta$; en consecuencia

$$\int_1^{\cosh t}\frac{dx}{\sqrt{x^2-1}} = \int_0^t\frac{\operatorname{senh}\theta\,d\theta}{\sqrt{\cosh^2\theta-1}} = \int_0^t\frac{\operatorname{senh}\theta}{|\operatorname{senh}\theta|}d\theta = \cdots$$

$\cosh^2\theta - \operatorname{senh}^2\theta = 1$ y $t \geq 0$

(k) $I = \int_0^2\frac{dx}{\sqrt{|x-1|}}$

Observemos que $f(x) = \frac{1}{\sqrt{|x-1|}}$ es continua en $[0,1)\cup(1,2]$, entonces

$$I = \lim_{\alpha\to 1^-}\int_0^\alpha\frac{dx}{\sqrt{1-x}} + \lim_{\alpha\to 1^+}\int_\alpha^2\frac{dx}{\sqrt{x-1}} = \lim_{\alpha\to 1^-}2\left(1-\sqrt{1-\alpha}\right) + \lim_{\alpha\to 1^+}2\left(1-\sqrt{\alpha-1}\right) = \cdots$$

$x-1<0 \ \forall\, 0\leq x<1 \qquad x-1>0 \ \forall\, 1<x\leq 2$

4. Determinar si las integrales siguientes convergen o divergen.

(a) $I = \int_1^\infty\frac{2x^2+1}{x^4+2x+1}dx$.

Observemos que $x \geq 1$, entonces $x^4+2x+1 \geq x^4 \Rightarrow \frac{2x^2+1}{x^4+2x+1} \leq \frac{2}{x^2}+\frac{1}{x^4}$

$$\Rightarrow I \leq \int_1^\infty\left(\frac{2}{x^2}+\frac{1}{x^4}\right)dx = 2\int_1^\infty\frac{dx}{x^2} + \int_1^\infty\frac{dx}{x^4}\ \cdots$$

(b) $I = \int_2^\infty\frac{dx}{(1+x)\log x}$

Tenemos que $\log x > 1-x \ \forall x \geq 2 \Rightarrow \frac{1}{(1+x)\log x} < \frac{1}{(1+x)(1-x)} = \frac{1}{2}\frac{1}{1-x}+\frac{1}{2}\frac{1}{1+x}$

$$\Rightarrow I \leq \lim_{\alpha\to\infty}\left[\int_2^\alpha\frac{dx}{2(1-x)} + \int_2^\alpha\frac{dx}{2(1+x)}\right] = \lim_{\alpha\to\infty}\frac{1}{2}\log\frac{1+\alpha}{\alpha}\frac{1}{2} = \cdots$$

(c) $I = \int_1^\infty\frac{dx}{1+x^{1/2}}dx$

Usar criterio de comparación con $g(x) = \frac{1}{x^{1/2}}$.

(d) $I = \int_2^\infty\frac{\sqrt{x^2+1}}{x}dx$

Observemos que $f(x) = \frac{\sqrt{x^2+1}}{x} = \sqrt{1+\frac{1}{x^2}} \xrightarrow{x\to\infty} 1\ \ldots$

(e) $I = \int_1^\infty\frac{x}{e^x-1}dx$

Notemos que $e^x > 1+x^3 \ \forall x \geq 1 \Rightarrow \frac{x}{e^x-1} < \frac{1}{x^2} \Rightarrow I < \lim_{\alpha\to\infty}\int_1^\infty\frac{dx}{x^2}\ \cdots$

(f) $I = \int_0^\infty\frac{\arctan x}{1+x^4}dx$.

Observemos que $\arctan < \frac{\pi}{2} \ \forall x \geq 0$, entonces $\frac{\arctan x}{1+x^4} < \frac{\pi}{2}\frac{1}{1+x^4} < \frac{\pi}{2}\frac{1}{1+x^2}$

$$\Rightarrow I \leq \lim_{\alpha\to\infty}\int_0^\alpha\frac{\pi}{2}\frac{dx}{1+x^2} = \frac{\pi}{2}\lim_{\alpha\to\infty}\arctan\alpha = \cdots$$

(g) $I = \int_0^\infty e^{-x^2}dx$.

Notemos que $x^2-2x+1 = (x+1)^2 > 0 \ \forall x \geq 0$, entonces $-x^2 < -2x+1$; dado que la función exponencial es monótona creciente, entonces

$$e^{-x^2} < e\,e^{-2x} \Rightarrow I < \lim_{\alpha\to\infty}e\int_0^\alpha e^{-2x}dx = \cdots$$

(i) $I = \int_0^1 e^{1/x}dx$

Sea $u = \frac{1}{x}$, entonces $du = -\frac{dx}{x^2} \Rightarrow I = \int_\infty^1 -\frac{1}{u^2}e^u du = \lim_{\alpha\to\infty}\int_1^\alpha\frac{e^u}{u^2}du$

Observemos que $\lim_{u\to\infty}\frac{e^u}{u^2} \underset{L'H}{=} \lim_{u\to\infty}\frac{e^u}{2u} \underset{L'H}{=} \lim_{u\to\infty}\frac{e^u}{2}$ no existe...

(h) $I = \int_0^1\frac{\operatorname{sen}x}{\sqrt{x}}dx$.

Veamos que $\frac{\operatorname{sen}x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \ \forall\, 0<x\leq 1 \Rightarrow I \leq \lim_{\alpha\to 0^+}\int_\alpha^1\frac{dx}{\sqrt{x}} = \cdots$

(j) $I = \int_0^\infty\frac{|\operatorname{sen}x|}{x^{3/2}}dx$

Observemos que $I = \underbrace{\int_0^1\frac{|\operatorname{sen}x|}{x^{3/2}}dx}_{A} + \underbrace{\int_1^\infty\frac{|\operatorname{sen}x|}{x^{3/2}}dx}_{B}$

Tenemos que $\frac{|\operatorname{sen}x|}{x^{3/2}} = \frac{|\operatorname{sen}x|}{|x|}\frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \ \forall\, 0<x\leq 1. \Rightarrow A \leq \lim_{\alpha\to 0^+}\int_\alpha^1\frac{dx}{\sqrt{x}}\ \cdots$

Por el otro lado, $\frac{|\operatorname{sen}x|}{x^{3/2}} \leq \frac{1}{x^{3/2}} \ \forall x \geq 1 \Rightarrow B \leq \lim_{\alpha\to\infty}\int_1^\alpha\frac{dx}{x^{3/2}}\ \cdots$

(k) $I = \int_3^\infty\frac{\log x}{(x-3)^4}dx$

Observemos que $I = \int_3^4\frac{\log x}{(x-3)^4}dx + \int_4^\infty\frac{\log x}{(x-3)^4}dx$. De esta manera, tenemos que $\log x > 1 \ \forall x \geq 4$, entonces $\int_4^\infty\frac{\log x}{(x-3)^4}dx < \lim_{\alpha\to\infty}\int_4^\alpha\frac{dx}{(x-3)^4} = \cdots$

Por el otro lado, notemos que si $u = x-3$, entonces

$$\int_3^4\frac{\log x}{(x-3)^4}dx = \int_0^1\frac{\log(u+3)}{u^4}du$$

Notemos que $\log(\cdot)$ es monótono creciente, entonces $\log(u+3) > \log u$. Como $\log u > u \ \forall\, 0<u<1$, entonces $\frac{\log(u+3)}{u} > 1 \Rightarrow \int_0^1\frac{\log(u+3)}{u^4}du > \int_0^1\frac{du}{u^3} = \cdots$