Laboratorio 14

1. Calcula:

(a) $\lim\limits_{x\to 0} \dfrac{1-\cancel{e^{-x^2}}^{\,0}}{\int_0^x x\,\cancel{e^{-t^2}}^{\,0}\,dt} = \lim\limits_{x\to 0} \dfrac{-\dfrac{e^{-x^2}-1}{x}}{\int_0^x e^{-t^2}dt} \overset{\text{T.F.C. L'H}}{=} \lim\limits_{x\to 0} \dfrac{-\dfrac{d^2 e^{-x^2}}{dx^2}}{e^{-x^2}} = -\dfrac{d}{dx}\left(-2x e^{-x^2}\right)\Big|_{x=0} = 2$ //

$$\boxed{\begin{gathered}\lim_{x\to 0}\frac{e^{-x^2}-e^{-0^2}}{x-0} = \left(e^{-x^2}\right)'\Big|_{x=0} = 0\\ \lim_{x\to 0}\int_0^x e^{-t^2}dx = \lim_{x\to 0} e^{-c^2}x = 0\\ \underbrace{\exists\, c\in[0,x]}_{\text{T.V.M}}\end{gathered}}$$

(b) $\lim\limits_{x\to\infty}\left(x\,3^{2/x}-x\right) = \lim\limits_{x\to\infty}\dfrac{3^{2/x}-1}{1/x} \overset{y=\frac{1}{x}}{=} \lim\limits_{y\to 0^+}\dfrac{3^y-3^0}{y-0}$.

2. Deducir $c\in\mathbb{R}$ tal que $\lim\limits_{x\to\infty}\left(\dfrac{cx}{cx+1}\right)^{x} = 9$.

$9 = \lim\limits_{x\to\infty}\left(\dfrac{1}{1+\frac{1}{cx}}\right)^x = \dfrac{1}{\lim\limits_{x\to\infty}\left(1+\frac{1/c}{x}\right)^x} = \dfrac{1}{e^{1/c}} \quad \therefore \quad c = \dfrac{1}{\log 9} = \dfrac{1}{2\log 3}$ //

3. Determinar:

(a) $\int \cos\sqrt{3x+4}\,dx = \int \frac{2}{3}u\cos u\,du = \dots$

$\begin{cases} u^2 = 3x+4 \\ 2u\,du = 3dx\end{cases}$ por partes

(b) $\int_0^{\log 2}\sqrt{1+\cos 2x}\,dx$ Usar $\cosh^2 x = \frac{1}{2}(1+\cosh 2x)$

(c) $\int\sqrt{e^{2x}-9}\,dx = \int\sqrt{3^2u^2-3^2}\,\dfrac{du}{3u} = \int\dfrac{\sqrt{u^2-1}}{u}du = \int\dfrac{\sqrt{\cosh^2\theta-1}}{\cosh\theta}\operatorname{senh}\theta\,d\theta =$

$\begin{cases} u = 3e^x \\ dx = \frac{1}{3}\frac{du}{u}\end{cases}$ $\begin{cases} u=\cosh\theta \\ du = \operatorname{senh}\theta\,d\theta\end{cases}$; $\cosh^2\theta-\operatorname{senh}^2\theta = 1$

identidad del doble ángulo.

$= \int\dfrac{|\operatorname{senh}\theta|\operatorname{senh}\theta}{\cosh\theta}d\theta = \pm\int\dfrac{\operatorname{senh}^2\theta}{\cosh\theta}d\theta = \pm\int\cosh\theta\,d\theta \mp \int\operatorname{sech}\theta\,d\theta =$

$= \pm\operatorname{senh}\left(\operatorname{arcosh}3e^x\right) \mp$ (Buscar dos funciones donde su derivada involucre a la $\operatorname{sech}\theta$. + multiplicar $\operatorname{sech}\theta$ por $\frac{A+B}{A+B}$ $\sim \log(\dots)$)

(d) $\int\sqrt{4x^2+4x+10}\,dx$

Observemos que $\sqrt{4x^2+4x+10} = \sqrt{(2x)^2+2\cdot 2x+4+6} = \sqrt{(2x+2)^2+(\sqrt{6})^2}$.

Sea $2x+2 = \sqrt{6}\operatorname{senh}\theta \Rightarrow dx = \frac{\sqrt{6}}{2}\cosh\theta\,d\theta$, entonces

$\int\sqrt{4x^2+4x+10}\,dx = \int\sqrt{(\sqrt{6}\operatorname{senh}\theta)^2+(\sqrt{6})^2}\,\frac{\sqrt{6}}{2}\cosh\theta\,d\theta = 3\int\cosh^2\theta\,d\theta = \dots$

- por partes
- definición $\cosh\theta$
- indentidad de doble ángulo.

(e) $\int\dfrac{x^4+2x^2-2}{x^3+2x}dx$ Hacer división sintética y considerar $\dfrac{1}{x^3+2x} = \dfrac{A}{x}+\dfrac{Bx+C}{x^2+2}$

(f) $\int\dfrac{4e^{-x}}{2-e^{-x}}dx$ Considerar $u=e^x$ y fracciones parciales.

4. Calcular

(a) $I = \int_{-\infty}^{\infty}\dfrac{e^x}{e^{2x}+9}dx = \lim\limits_{\alpha\to\infty}\int_{-\alpha}^{\alpha}\dfrac{e^x}{e^{2x}+3^2}dx = \dfrac{1}{3}\lim\limits_{\alpha\to\infty}\int_{3e^{-\alpha}}^{3e^{\alpha}}\dfrac{du}{u^2+1} = \dfrac{1}{3}\lim\limits_{\alpha\to\infty}\left[\underbrace{\arctan 3e^{\alpha}}_{\to\pi/2}-\underbrace{\arctan 3e^{-\alpha}}_{\to 0}\right]$

$\begin{cases} u = 3e^x \\ du = 3e^x dx\end{cases}$

(b) $I = \int_1^{\infty}\dfrac{dx}{e^x+e^{2x}}$ Sea $u=e^x$ y fracciones parciales para calcular su valor.

(c) $I = \int_2^4\dfrac{dx}{\sqrt{4x-x^2}}$. Notemos que $\sqrt{4x-x^2} = \sqrt{-(x^2-4x)} = \sqrt{-(x^2-2\cdot 2x+4-4)} =$

$= \sqrt{-(x-2)^2+4} = \sqrt{2^2-(x-2)^2}$

Sea $x-2 = 2\operatorname{sen}\theta \Rightarrow dx = 2\cos\theta\,d\theta \Rightarrow \sqrt{2^2-(x-2)^2} = 2\sqrt{1-\operatorname{sen}^2\theta} = 2|\cos\theta|$

$2-2 = 2\operatorname{sen}0$ y $4-2 = 2\operatorname{sen}\frac{\pi}{2}$, entonces

$I = \int_0^{\pi/2}\dfrac{2\cos\theta\,d\theta}{2|\cos\theta|} = \int_0^{\pi/2}d\theta = \dfrac{\pi}{2}$ //

5. Determinar los valores de $a>0$ y $c\in\mathbb{R}$ tales que $I = \int_1^{\infty}\dfrac{(c-a)x+a}{x(2x+a)}dx = 1$.

Existen $A, B\in\mathbb{R}$ tales que

$\dfrac{(c-a)x+a}{x(2x+a)} = \dfrac{A}{x}+\dfrac{B}{2x+a} = \dfrac{2xA+aA+Bx}{x(2x+a)} = \dfrac{(2A+B)x+aA}{x(2x+a)} \Rightarrow$ $A=1$, $B = c-a-2$

$\Rightarrow \dfrac{(c-a)x+a}{x(2x+a)} = \dfrac{1}{x}+\dfrac{c-a-2}{2x+a}$

$\Rightarrow 1 = \lim\limits_{\alpha\to\infty}\int_1^{\alpha}\left(\dfrac{1}{x}-\dfrac{c-a-2}{2x+a}\right)dx = \lim\limits_{\alpha\to\infty}\left(\log|\alpha|-\dfrac{c-a-2}{2}\log\left|\dfrac{2\alpha+a}{2+a}\right|\right) =$

$= \lim\limits_{\alpha\to\infty}\log\left|\dfrac{2\alpha+a}{(2+a)\alpha}\right|^{\frac{c-a-2}{2}} = \log\left|\dfrac{2}{2+a}\right|^{\frac{c-a-2}{2}} \Rightarrow \left|\dfrac{2}{2+a}\right|^{\frac{c-a-2}{2}} = e$

Sea $c=a$, entonces $\dfrac{2}{2+a} = \dfrac{1}{e} \Rightarrow \dots$ //

6. Determinar si las integrales siguientes convergen o no:

(a) $I = \int_4^{\infty}\dfrac{1}{\log x^3-1}dx$. Observemos que $x>\log x \;\forall x\geq 4 \Rightarrow 3x-1 > \log x^3-1$

$\Rightarrow \dfrac{1}{\log x^3-1} > \dfrac{1}{3x-1} \Rightarrow \lim\limits_{\alpha\to\infty}\int_4^{\alpha}\dfrac{dx}{3x-1} < \lim\limits_{\alpha\to\infty}\int_4^{\alpha}\dfrac{dx}{\log x^3-1}$

(b) $I = \int_1^{\infty}\dfrac{dt}{e^{3t}-e^{-3t}}$ Tenemos que $I = \int_3^{\infty}\dfrac{du}{3(e^u-e^{-u})}$. Ahora observemos que

$\begin{cases} 3t = u \\ dt = \frac{1}{3}du\end{cases}$

- $\dfrac{1}{e^u-e^{-u}} = \dfrac{1}{\frac{e^u-e^{-u}}{2}}\dfrac{1}{2} = \dfrac{1}{2}\dfrac{1}{\operatorname{senh}u} = \dfrac{1}{2}\operatorname{csch}u \Rightarrow I = \int_3^{\infty}\operatorname{csch}u\,du =$
- $\dfrac{1}{e^u-e^{-u}} = \dfrac{e^u}{e^{2u}-1}$ y considerar $v = e^u \Rightarrow dv = e^u du \Rightarrow \dfrac{du}{e^u-e^{-u}} = \dfrac{dv}{v^2-1}$;

finalmente usar un cambio de variable hiperbólico o trigonométrico o usar fracciones parciales.

(c) $I = \int_0^1\dfrac{\arcsen x}{\sqrt{x}}dx$ Sabemos que $0\leq x\leq 1$, entonces si $x = \operatorname{sen}y \Rightarrow 0\leq\operatorname{sen}y\leq 1$

$\Rightarrow \arcsen 0\leq y\leq\arcsen 1$ i.e.: $0\leq y\leq\frac{\pi}{2} \Rightarrow 0\leq\dfrac{\arcsen x}{\sqrt{x}}\leq\dfrac{\pi}{2}\dfrac{1}{\sqrt{x}} = \dfrac{M}{(x-a)^{\alpha}}$,

$\Rightarrow 0\leq I\leq\lim\limits_{a\to 0^+}\int_a^1\dfrac{\pi}{2}\dfrac{dx}{\sqrt{x}}\dots$ donde $M=\frac{\pi}{2}$, $a=0$ y $\alpha=\frac{1}{2}$ //

(d) $I = \int_0^1\dfrac{\sec^2 x}{x\sqrt{x}}dx \leftarrow \dfrac{\sec^2 x}{x^{3/2}}$ y usar criterio del cociente.

(e) $I = \int_0^{\infty}\dfrac{\arctan x}{x^{3/2}}dx = \underbrace{\boxed{\int_0^1\dfrac{\arctan x}{x^{3/2}}dx}}_{A} + \underbrace{\boxed{\int_1^{\infty}\dfrac{\arctan x}{x^{3/2}}dx}}_{B}$ i.e.: $I = A+B$

Caso B:

Sea $f(x) = \dfrac{\arctan x}{x^{3/2}}$ y $g(x) = \dfrac{1}{x^{3/2}}$, donde $\int_1^{\infty}g(x)dx$ existe, entonces:

$\lim\limits_{x\to\infty}\left|\dfrac{f(x)}{g(x)}\right| = \lim\limits_{x\to\infty}|\arctan x| = \dfrac{\pi}{2} \quad\therefore$ B existe dado el criterio del cociente.

Caso A:

Considerando el criterio usado anteriormente, tenemos que $\int_0^1 g(x)dx = \int_0^1\frac{dx}{x^{3/2}}$ diverge.

Por lo tanto, A diverge y consecuentemente. I //

7. (a) Calcula, si existe, el límite

$\lim\limits_{x\to\infty}\cancel{x^2}\cancel{\arctan\frac{1}{x^2}} = \boxed{\lim\limits_{x\to\infty}\dfrac{\arctan\frac{1}{x^2}}{1/x^2}} \overset{y=\frac{1}{x^2}}{=} \lim\limits_{y\to 0^+}\dfrac{\arctan y}{y} = 1$ //

L'H / definición de derivada

(b) Usa el inciso anterior para estudiar la naturaleza de la siguiente integral

$$I = \int_1^{\infty}\arctan\frac{1}{x^2}dx$$

Sea $f(x) = \arctan\frac{1}{x^2}$ y $g(x) = \frac{1}{x^2}$, donde $\int_1^{\infty}g(x)dx$ existe y por el inciso anterior, tenemos que $\lim\limits_{x\to\infty}\left|\dfrac{f(x)}{g(x)}\right| = \lim\limits_{x\to\infty}x^2\arctan\frac{1}{x^2} = 1$; por lo tanto, $I<\infty$ //