Ejercicios-ejemplo extra

Ej 1 Evaluar los siguientes límites

(a) $\lim_{x\to\infty} x\left[\left(1+\frac{1}{x}\right)^x - e\right] = \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^x - e}{1/x} \underset{L'H}{=} \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^x \log\left(1+\frac{1}{x}\right) - \frac{\left(1+\frac{1}{x}\right)^x}{1+x}}{-1/x^2} =$

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

$f(x) = \left(1+\frac{1}{x}\right)^x$

$\Rightarrow \frac{f'(x)}{f(x)} = \log\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

$= \left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)\left[\lim_{x\to\infty}\left(\frac{1}{1+x} - \log\left(1+\frac{1}{x}\right)\right)\right]\lim_{x\to\infty} x^2$

Material adicional (corrección).

Notemos que como el factor $\left(1+\frac{1}{x}\right)^x \xrightarrow{x\to\infty} e$, entonces es suficiente calcular

$\lim_{x\to\infty} \frac{\frac{1}{1+x} - \log\left(1+\frac{1}{x}\right)}{1/x} \underset{L'H}{=} \lim_{x\to\infty} \frac{-\frac{1}{(1+x)^2} + \frac{1}{x(x+1)}}{-2/x^3} = \lim_{x\to\infty} \frac{1}{2}\left(\frac{x^3}{(1+x)^2} - \frac{x^3}{x(x+1)}\right) =$

$= \lim_{x\to\infty} \frac{1}{2}\left[\frac{1}{(1/x+1)^2} - \frac{1}{1+1/x}\right]x = \frac{1}{2}\lim_{x\to\infty}\frac{-1/x}{(1+1/x)^2}\,x = -\frac{1}{2}$. Por lo tanto,

concluimos que $\lim_{x\to\infty} x\left[\left(1+\frac{1}{x}\right)^x - e\right] = -\frac{e}{2}$

(b) $\lim_{x\to 0}\left(\frac{\operatorname{sen} x}{x}\right)^{1/x^2}$

Consideremos el límite

$L = \lim_{x\to 0}\frac{\log\left(\frac{\operatorname{sen} x}{x}\right)}{x^2} \underset{L'H}{=} \lim_{x\to 0}\frac{\frac{\cos x}{\operatorname{sen} x} - \frac{1}{x}}{2x} = \lim_{x\to 0}\frac{x\cos x - \operatorname{sen} x}{2x^2\operatorname{sen} x} \underset{L'H}{=}$

$= \lim_{x\to 0}\frac{-x\operatorname{sen} x}{4x\operatorname{sen} x + 2x^2\cos x} \underset{L'H}{=} \lim_{x\to 0}\frac{-\cos x}{6\cos x - 2x\operatorname{sen} x} = -\frac{1}{6} \quad \therefore \lim_{x\to 0}\left(\frac{\operatorname{sen} x}{x}\right)^{1/x^2} = e^{-1/6}$

Ej 2. Determinar los valores de a y b de tal modo que

$$L = \lim_{x\to 0}\frac{1}{ax - \operatorname{sen} x}\int_0^x \frac{u^2}{\sqrt{b+u}}\,du = 1.$$

Notemos que por la Regla de L'Hopital y el T.F.C., se tiene que

$L \underset{L'H}{=} \lim_{x\to 0}\frac{\frac{x^2}{\sqrt{b+x}}}{a-\cos x} \underset{\substack{a=1\\L'H}}{=} \lim_{x\to 0}\frac{\frac{2x\sqrt{b+x} - \frac{x^2}{2\sqrt{b+x}}}{b+x}}{\operatorname{sen} x} \underset{L'H}{=} \lim_{x\to 0}\frac{2\sqrt{b+x} - \frac{x}{2\sqrt{b+x}}}{(b+x)\frac{\operatorname{sen} x}{x}} = \frac{2\sqrt{b}}{b}$

$\Rightarrow b = 4$

Ej 3 Calcular $\lim_{w_1\to w_2} y(t)$, donde $y(t) = \frac{g}{w_1^2 - w_2^2}(\operatorname{sen} w_1 t - \operatorname{sen} w_2 t)$.

$w_1^2 - w_2^2 = (w_1 - w_2)(w_1 + w_2)$

Tenemos que $y(t) = \frac{\operatorname{sen} w_1 t - \operatorname{sen} w_2 t}{w_1 t - w_2 t}\,\frac{gt}{w_1 + w_2} \longrightarrow \frac{gt}{2w_2}\cos w_2 t$

Ej 4. Probar que $(-1)^n\int_{-1}^{1}(x^2-1)^n\,dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$

Observemos que

$I_n = (-1)^n\int_{-1}^{1}(x^2-1)^n\,dx = 2\int_0^1(1-x^2)^n\,dx = 2\int_0^{\pi/2}\operatorname{sen}^{2n}\theta\,\operatorname{sen}\theta\,d\theta =$

(función par; $x = \cos\theta$, $dx = -\operatorname{sen}\theta\,d\theta$; $\operatorname{sen}^{2n}\theta\operatorname{sen}\theta = \operatorname{sen}^{2n+1}\theta$)

$\begin{cases} f(\theta) = \operatorname{sen}^{2n}\theta \\ g'(\theta) = \operatorname{sen}\theta \end{cases} \Rightarrow \begin{matrix} f'(\theta) = 2n\operatorname{sen}^{2n-1}\theta\cos\theta \\ g(\theta) = -\cos\theta \end{matrix}$

$= -2\operatorname{sen}^{2n}\theta\cos\theta\Big|_0^{\pi/2} + 4n\int_0^{\pi/2}\operatorname{sen}^{2n-1}\theta\cos^2\theta\,d\theta = 4n\int_0^{\pi/2}\operatorname{sen}^{2n-1}\theta\,d\theta - 4n\int_0^{\pi/2}\operatorname{sen}^{2n+1}\theta\,d\theta$

$\Rightarrow I_n = 2\int_0^{\pi/2}\operatorname{sen}^{2n+1}\theta\,d\theta = 2nI_{n-1} - 2nI_n \Rightarrow I_n = \frac{2n}{2n+1}I_{n-1} \quad \forall n\in\mathbb{N}$

En consecuencia, tenemos que

$I_n = \frac{2n}{2n+1}I_{n-1} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}I_{n-2} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}\frac{2n-4}{2n-3}I_{n-3} = \ldots =$

$= \frac{2n}{2n+1}\frac{2n-2}{2n-1}\frac{2n-4}{2n-3}\cdots\frac{4}{3}\cdot\frac{2}{1} = \frac{2n}{2n+1}\frac{2n}{2n}\frac{2n-2}{2n-1}\frac{2n-2}{2n-2}\frac{2n-4}{2n-3}\cdots\frac{4}{4}\frac{4}{3}\frac{2}{2}\frac{2}{1} =$

$= \frac{1}{(2n+1)!}2^2n^2\,2^2(n-1)^2\,2^2(n-2)^2\cdots 2^2 2^2\cdot 2^2\cdot 1^2 = 2^{2n+1}(n!)^2$

$n! = n(n-1)\cdots 3\cdot 2\cdot 1$

$\therefore (-1)^n\int_{-1}^{1}(x^2-1)^n\,dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$

Ej 5 Sea $f''(x)$ una función continua tal que $\int_0^\pi (f(x)+f''(x))\operatorname{sen} x\,dx = 2$. Considerando que $f(\pi) = 1$, encontrar el valor de $f(0)$.

Observemos que:

$I_1 = \int_0^\pi f(x)\operatorname{sen} x\,dx = f(x)\cos x\Big|_\pi^0 + \int_0^\pi f'(x)\cos x\,dx = f(0) + f(\pi) + \int_0^\pi f'(x)\cos x\,dx$

$\begin{cases} u(x) = f(x) \\ v'(x) = \operatorname{sen} x \end{cases} \Rightarrow \begin{matrix} u'(x) = f'(x) \\ v(x) = -\cos x \end{matrix}$

$I_2 = \int_0^\pi f''(x)\operatorname{sen} x\,dx = f'(x)\operatorname{sen} x\Big|_0^\pi - \int_0^\pi f'(x)\cos x\,dx$

$\begin{cases} u(x) = \operatorname{sen} x \\ v'(x) = f''(x) \end{cases} \Rightarrow \begin{matrix} u'(x) = \cos x \\ v(x) = f'(x) \end{matrix}$

$\Rightarrow 2 = \int_0^\pi (f(x)+f''(x))\operatorname{sen} x\,dx = I_1 + I_2 = f(0) + 1 + \int_0^\pi f'(x)\cos x\,dx - \int_0^\pi f'(x)\cos x\,dx =$

$= f(0) + 1 \quad \therefore f(0) = 1$

Ej 6 Hallar el valor de $f'(0)$, si $f(x) = \begin{cases} \frac{g(x)}{x}, & x\neq 0 \\ 0, & x = 0 \end{cases}$ y $g(0) = 0 = g'(0)$ y $g''(0) = \alpha\in\mathbb{R}$.

Por definición tenemos que

$f'(0) = \lim_{h\to 0}\frac{1}{h}(f(h) - f(0)) = \lim_{h\to 0}\frac{g(h)}{h^2} \underset{L'H}{=} \lim_{h\to 0}\frac{g'(h)}{2h} \underset{L'H}{=} \lim_{h\to 0}\frac{g''(h)}{2} = \frac{\alpha}{2}$

Ej 7. Producto de Wallis.

Sea $n>1$, entonces $\int\operatorname{sen}^n u\,du = -\frac{1}{n}\cos u\operatorname{sen}^{n-1}u + \frac{n-1}{n}\int\operatorname{sen}^{n-2}u\,du$ como consecuencia del método de integración por partes y la identidad $\operatorname{sen}^2 u + \cos^2 u = 1$ $\forall u\in\mathbb{R}$.

De este modo, observemos que $\int_0^{\pi/2}\operatorname{sen}^n u\,du = \frac{n-1}{n}\int_0^{\pi/2}\operatorname{sen}^{n-2}u\,du$

Ahora, consideremos los casos:

(i) $n = 2k$, con $k\in\mathbb{N}$.

$\int_0^{\pi/2}\operatorname{sen}^{2k}u\,du = \frac{2k-1}{2k}\frac{2k-3}{2k-2}\cdots\frac{1}{2}\int_0^{\pi/2}du = \frac{2k-1}{2k}\frac{2k-3}{2k-2}\cdots\frac{1}{2}\frac{\pi}{2}$

(ii) $n = 2k+1$, con $k\in\mathbb{N}$:

$\int_0^{\pi/2}\operatorname{sen}^{2k+1}u\,du = \frac{2k}{2k+1}\frac{2k-2}{2k-1}\cdots\frac{2}{3}\int_0^{\pi/2}\operatorname{sen} u\,du = \frac{2k}{2k+1}\frac{2k-2}{2k-1}\cdots\frac{2}{3}\cdot 1$

Al calcular el cociente de los incisos anteriores tenemos que

$$\frac{\int_0^{\pi/2}\operatorname{sen}^{2k}u\,du}{\int_0^{\pi/2}\operatorname{sen}^{2k+1}u\,du} = \frac{2k-1}{2k}\frac{2k+1}{2k}\cdots\frac{3}{4}\frac{5}{4}\frac{1}{2}\frac{3}{2}\frac{\pi}{2}$$

De este modo, tenemos que

$$(*) \qquad \frac{\pi}{2} = \frac{2k\cdot 2k\cdots 4\cdot 4\,2\cdot 2}{(2k+1)(2k-1)\cdots 5\cdot 3\cdot 3\cdot 1}\left[\frac{\int_0^{\pi/2}\operatorname{sen}^{2k}u\,du}{\int_0^{\pi/2}\operatorname{sen}^{2k+1}u\,du}\right]$$

Ahora, observemos que $0\le\operatorname{sen}^{2k+1}u\le\operatorname{sen}^{2k}u\le\operatorname{sen}^{2k-1}u$ para $k\in\mathbb{N}$, puesto que $0\le\operatorname{sen} u\le 1$ $\forall u\in[0,\pi/2]$; entonces, por la homogeneidad de la integral, tenemos que

$$0 < \int_0^{\pi/2}\operatorname{sen}^{2k+1}u\,du \le \int_0^{\pi/2}\operatorname{sen}^{2k}u\,du \le \int_0^{\pi/2}\operatorname{sen}^{2k-1}u\,du$$

$$\Rightarrow 1 \le \frac{\int_0^{\pi/2}\operatorname{sen}^{2k}u\,du}{\int_0^{\pi/2}\operatorname{sen}^{2k+1}u\,du} \le \frac{\int_0^{\pi/2}\operatorname{sen}^{2k-1}u\,du}{\int_0^{\pi/2}\operatorname{sen}^{2k+1}u\,du} = \frac{\frac{2k-2}{2k-1}\cdots\frac{2}{3}}{\frac{2k}{2k+1}\cdots\frac{2}{3}} = \frac{2k+1}{2k} = 1 + \frac{1}{2k}\xrightarrow{k\to\infty}1$$

En consecuencia, a partir de (*), obtenemos

$$\frac{\pi}{2} = \left(\lim_{k\to\infty}\frac{\int_0^{\pi/2}\operatorname{sen}^{2k}u\,du}{\int_0^{\pi/2}\operatorname{sen}^{2k+1}u\,du}\right)\left(\lim_{k\to\infty}\frac{2k}{2k+1}\right)\left(\underbrace{\lim_{k\to\infty}2k\frac{(2k-2)^2\cdots 2^2}{(2k-1)^2\cdots 3^2}}_{L^2}\right)$$

Por continuidad de la función raíz cuadrada, tenemos que

$$L = \lim_{k\to\infty}\frac{(2k-2)\cdots 2}{(2k-1)\cdots 3}\sqrt{2k} = \sqrt{\frac{\pi}{2}}$$

$$\Rightarrow \sqrt{\frac{\pi}{2}} = \lim_{k\to\infty}\frac{(2k-2)\cdots 2\sqrt{2k}}{(2k-1)\cdots 3}\cdot\frac{(2k-2)\cdots 4\cdot 2}{(2k-2)\cdots 4\cdot 2}\,\frac{(2k)^2}{2k\,2k} = \lim_{k\to\infty}\frac{(2k)^2(2k-2)^2\cdots 4^2\cdot 2^2}{2k(2k-1)(2k-2)\cdots 4\cdot 3\cdot 2}\cdot\frac{\sqrt{2k}}{2k} =$$

$$= \lim_{k\to\infty}\frac{2^{2k}[k\cdot(k-1)\cdots 2]^2}{(2k)!\,\sqrt{2k}} = \lim_{k\to\infty}\frac{2^{2k}(k!)^2}{(2k)!\,\sqrt{2k}}$$

$$\therefore \lim_{k\to\infty}\frac{2^{2k}(k!)^2}{(2k)!\,\sqrt{k}} = \sqrt{\pi}$$