The restricted three body problem on surfaces of constant curvature

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Abstract

We consider a symmetric restricted three-body problem on surfaces $M^2_\kappa$ of constant Gaussian curvature $\kappa \neq 0$, which can be reduced to the cases $\kappa = \pm 1$. This problem consists in the analysis of the dynamics of an infinitesimal mass particle attracted by two primaries of identical masses describing elliptic relative equilibria of the two body problem on $M^2_\kappa$, i.e., the primaries move on opposite sides of the same parallel of radius $a$. The Hamiltonian formulation of this problem is pointed out in intrinsic coordinates. The goal of this paper is to describe analytically, important aspects of the global dynamics in both cases $\kappa = \pm 1$ and determine the main differences with the classical Newtonian circular restricted three-body problem. In this sense, we describe the number of equilibria and its linear stability depending on its bifurcation parameter corresponding to the radial parameter $a$. After that, we prove the existence of families of periodic orbits and KAM 2-tori related to these orbits.

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1. Introduction

A first approach of the N-body problem on non Euclidean spaces was raised simultaneously by N. Lovachevsky [20], and by J. Bolyai [3] in the 1830’s. In 1860 P.J. Serret extended the gravitational force to the sphere \( S^2 \) and solved the corresponding Kepler problem [28]. In 1885, W. Killing adapted the Bolyai–Lobachevsky gravitational law to \( S^3 \) and introduced the cotangent potential as a good extension of the classical potential [16].

One of the fundamental results on the area is due to H. Liebmann, who in 1902 proved that the orbits of the curved Kepler problem are indeed conics in \( S^3 \) and \( H^3 \) and a bit later, he proved an important result corresponding to an \( S^2 \)- and \( H^2 \)-analogues Bertrand’s theorem, which states that there exist only two analytic potentials in the Euclidean space such that all bounded orbits are closed (see [18,19]).

Relevant and more recent works correspond to those obtained by V. Kozlov [17] and, particularly, contributions to the two-dimensional case of the Kepler problem belong to J. Cariñena, M. Rañada and M. Santander [4], who provided a unified formulation of the Kepler problem for an arbitrary constant curvature \( \kappa \), getting conics orbits depending on the curvature parameter, showing that the well known conics orbits in Euclidean spaces can be extended to spaces of constant curvature. A similar approach of the two dimensional curved Kepler problem, but with emphasis on the dynamics, was treated in [1].

The curved N-body problem for \( N \geq 2 \) was introduced by F. Diacu et al. in a couple of papers The N-body problem in spaces of constant curvature. Parts I and II [11], [12], where a clear and detailed deduction of the equations of motion for any number \( N \geq 2 \) of bodies is presented, using variational methods of the theory of constrained Lagrangian dynamics, getting the equations of motion depending on the curvature \( \kappa \). Some other interesting results in this direction can be found in [8], [9], [6], a complete historical background is given in [7].

We know that all 2-dimensional spaces of constant curvature \( \kappa \) are characterized by the sign of the curvature as the surfaces

\[
\mathbb{M}_\kappa^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + \sigma z^2 = \kappa^{-1}\},
\]

with \( \sigma = \text{sign}(\kappa) \), which induces to define the inner product

\[
(a_x, a_y, a_z) \odot (b_x, b_y, b_z) := a_x b_x + a_y b_y + \sigma a_z b_z.
\]

If \( \kappa > 0 \), then the surface is the 2 dimensional sphere of radius \( R = 1/\sqrt{\kappa} \) denoted by \( S^2_\kappa \) embedded in the Euclidean space \( \mathbb{R}^3 \). If \( \kappa = 0 \), we recover the Euclidean space \( \mathbb{R}^2 \). If \( \kappa < 0 \), then the surface is the upper sheet of the hyperboloid \( x^2 + y^2 - z^2 = \kappa^{-1} \), with \( z > 0 \), denoted by \( \mathbb{H}^2_\kappa \), which corresponds to a surface embedded in the 3 dimensional Minkowski space \( \mathbb{R}^{2,1} \) (\( \mathbb{R}^3 \) endowed with the Lorentz inner product).

According to [11] the equations of motion for \( N \geq 2 \) bodies on a surface of curvature \( \kappa \) are given by
The velocity of \( \mathbf{q}_i \) is given by

\[
\ddot{\mathbf{q}}_i = \sum_{j=1, \ j \neq i}^{N} \frac{m_j |\kappa|^{3/2} \left[ \mathbf{q}_j - \sigma (\kappa \mathbf{q}_i \odot \mathbf{q}_j) \mathbf{q}_i \right]}{\left[ \sigma - \sigma (\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2 \right]^{3/2}} - \sigma (\mathbf{q}_i \odot \ddot{\mathbf{q}}_i) \mathbf{q}_i,
\]

with the constraints

\[
\mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}, \quad \mathbf{q}_i \odot \dot{\mathbf{q}}_i = 0, \quad i = 1, \cdots, N, \quad \sigma = \text{sgn}(\kappa).
\]

We observe that for \( \kappa \neq 0 \), equation (1) can be written independently of the curvature by means of a rescaling \( \frac{d\zeta}{dt} = |\kappa|^{3/4} \) and the change of coordinates \( \mathbf{q}_i \mapsto |\kappa|^{-1/2} \mathbf{q}_i \), so we will consider along this work that \( \kappa = \sigma = \pm 1 \).

The restricted three-body problem was introduced by L. Euler [14]. In his work, he considered three mutually interacting masses \( m_1, m_2, m_3 \), where \( m_3 \) is negligible so that it does not have influence in the motion of \( m_1 \) and \( m_2 \), but its dynamics is entirely determined by the motion of them. This problem can be extended naturally to the restricted \((N + 1)\)-body problem with \( N \) positive masses and the last one of negligible mass.

In order to define the restricted three-body problem on a surface of constant curvature \( \kappa = \pm 1 \), we start by considering an elliptic relative equilibrium of the curved 2-body problem, that is, a solution where the primaries are rotating uniformly with constant angular velocity \( \omega \) on fixed parallels of \( \mathbb{S}^2_1 \) or \( \mathbb{H}^2_{-1} \) (this type of solutions will be described in detail in Section 2). As usual we normalized the masses as \( m_1 = \mu, \ m_2 = 1 - \mu \) and assume that the primaries are moving on circles of radius \( r_1 \) and \( r_2 \), respectively. We call this problem the restricted three body problem on surfaces of constant curvature (R3BP on \( \mathbb{S}^2 \) or \( \mathbb{H}^2 \) by short).

In this article we study different aspects of the dynamics of the curved R3BP for the particular case in which the primaries move on the same circle, that is, when \( r_1 = r_2 = a \).

We observe that the above problem depends on two parameters \((\mu, r)\), where \( r \) is either \( r_1 \) or \( r_2 \). When \( r_1 \neq r_2 \) and consequently \( \mu \neq 1/2 \) the analysis results really cumbersome, in fact, the characterization of equilibria becomes hard to carry out by means of analytical methods as well as their stability. This is clearly evidenced in [22], where the analysis of this problem had to be complemented with numerical simulations. Due to this difficulty, we will consider essentially the symmetric problem, that is, when \( r_1 = r_2 = a \), where \( a \in (0, 1) \) if \( \kappa = 1 \); and \( a \in (0, \infty) \) if \( \kappa = -1 \). We refer it as the symmetric R3BP on \( \mathbb{S}^2 \) or \( \mathbb{H}^2 \). Most part of our results concern to this case.

In order to avoid the natural constraints of this problem, by using the stereographic projection through the equatorial plane as in [26] and [10], we work with intrinsic coordinates on the curved plane, i.e., \( \mathbb{R}^2 \) (in the case of \( \mathbb{S}^2 \)) or the Poincaré Disk (in the case of \( \mathbb{H}^2 \)), both endowed with the metric induced by the stereographic projection. In both cases, we use synodic coordinates in the corresponding surface to write the equations of motion as a Hamiltonian system where the Hamiltonian function is given by

\[
\mathcal{H} = \frac{\Delta^2}{8} (p_u^2 + p_v^2) + \omega(vp_u - up_v) - U(u, v),
\]

here \((u, v)\) are the coordinates of the position of the infinitesimal mass particle, \( \omega \) is the angular velocity of the primaries in the synodic frame, \( \Delta = 1 + \sigma (u^2 + v^2) \) and \( U(u, v) \) is the potential written in intrinsic coordinates.

A preliminary study of this model was given by R. Martínez and C. Simó in [22]. In their work, they obtained the relative equilibria depending on the parameters \( \kappa \) and the mass ratio.
Theorem non-degeneracy case not tori for parameter or orbits and given the constant analytically with this form 2. As we have mentioned previously, we focus on the symmetric case ($\mu = 1/2$). First, we look for the relative equilibrium points, its number and its linear stability, which of course depend on their bifurcations points in the parameter space. We do an analytical and detailed analysis of the type of stability for each equilibrium point in each region determined by the parameters. With this information, and using the Hamiltonian structure of our problem, by one hand we prove the existence of families of periodic orbits surrounding the poles of $S^2$ and the vertex of $H^2$ for certain values of the parameter $a$ (in order to get these families we use the Lyapunov Center Theorem [23]). In the case of $S^2$ we also prove the existence of KAM 2-tori related to the above family of periodic orbits by using normal forms and Arnold’s Theorem for the isoenergetically non-degeneracy condition. On the other hand, using Averaging Theory in the Hamiltonian setting (see [27], [24], [23]), we prove the existence of a family of periodic orbits for sufficiently small values of the radial parameter $a$. Additionally, we study the existence of KAM two-dimensional tori related to these periodic orbits (see [2,24]). More precisely, we have used Han, Li and Yi’s Theorem (see details in [15]) that applies in the case of Hamiltonian systems with high-order proper degeneracy.

The paper is organized as follows. In Section 2 we define and characterize the elliptic relative equilibria. Section 3 is devoted to introduce the equations of motion and the Hamiltonian formulation in intrinsic coordinates, given by the stereographic projection through the equatorial plane $z = 0$. In Section 4, the number of relative equilibria for the symmetric problem is characterized in terms of the parameter $a = r_1 = r_2$, then we prove the existence of bifurcation values. In Section 5, we analyze the linear stability of the equilibrium points characterized in Section 4. In section 6, we prove the existence of a family of periodic orbits surrounding the poles of $S^2$ and the vertex of $H^2$, as well as, the existence of periodic orbits for the case when the radial parameter is small enough. Section 7 is devoted to the study of KAM 2-tori related to the periodic orbits studied in Section 6. Finally, we have added in the Appendix the coefficients of the normal form used to prove Theorem 7.1.

2. Relative equilibria in the 2-body problem on $S^2$ and $H^2$

We consider the motion of two particles of normalized masses $m_1 = \mu$ and $m_2 = 1 - \mu$, with $0 < \mu \leq \frac{1}{2}$ and positions $q_i = (x_i, y_i, z_i) \in \mathbb{M}_\sigma^2$. We shall assume $\sigma = 1$ or $\sigma = -1$, where $\mathbb{M}_1^2 = S^2$ and $\mathbb{M}_{-1}^2 = H^2$. According to (1) for $N = 2$ and $\kappa = \pm 1$, the equations of motion are given by
\[ \ddot{q}_1 = -\sigma (\dot{q}_1 \odot \dot{q}_1) q_1 + m_2 \frac{q_2 - \sigma (q_1 \odot q_2) q_1}{\sigma - \sigma (q_1 \odot q_2)^2} \frac{1}{\sqrt{T}}, \]

\[ \ddot{q}_2 = -\sigma (\dot{q}_2 \odot \dot{q}_2) q_2 + m_1 \frac{q_1 - \sigma (q_1 \odot q_2) q_2}{\sigma - \sigma (q_1 \odot q_2)^2} \frac{1}{\sqrt{T}}, \]

with the restrictions

\[ \sigma (q_i \odot q_i) = 1, \quad q_i \odot \dot{q}_i = 0, \quad i = 1, 2. \]

We observe that the above system has singularities at collision \((q_1 = q_2)\). Additionally, on \(S^2\) we also have antipodal singularities, that is when \((q_1 = -q_2)\). A complete analysis of the singularities of system (3) can be found in [12]. We start properly this paper with the definition of an elliptic relative equilibrium (see [11]).

**Definition 2.1.** An elliptic relative equilibrium is a solution of (3) of the form

\[ q_j = (r_j \cos(\omega t + \alpha_j), r_j \sin(\omega t + \alpha_j), z_j), \]

where \(z_j\) is a real number, with \(z_j \in (-1, 1)\) if \(\sigma = 1\) or \(z_j \in (1, \infty)\) if \(\sigma = -1\), \(\alpha_j \in [0, 2\pi]\), \(\omega \neq 0\) and \(r_j = \sqrt{\sigma - \sigma z_j^2}\).

We observe that each particle moves on a circle parallel to the \(x-y\)-plane (not necessarily of the same radius). This induces to introduce the useful notation

\[ q_j = r_j e^{i(\omega t + \alpha_j)} + z_j e_3, \quad \text{with } e_3 = (0, 0, 1). \]

Therefore, in the way to search elliptic relative equilibria we must find relations between the parameters \(z_1, z_2, m_1, m_2, \omega, \alpha_1\) and \(\alpha_2\), in order that (5) and (3) hold. The elliptic relative equilibria of the curved 2-body problem has been studied in [11] and [22], however, we find an explicit parametrization of the initial conditions generating this kind of solutions.

**Proposition 2.1.** Given \(0 < \mu \leq 1/2\). The elliptic relative equilibria must have initial conditions such that \(\alpha_1 = \alpha_2 + \pi\). For \(\sigma = -1\), there are exactly two elliptic relative equilibria, both with the same configuration, but with angular velocity \(\pm \omega(z_2, \mu)\), respectively. For \(\sigma = 1\), all elliptic relative equilibria are located on the same hemisphere, i.e., \(z_1z_2 > 0\) and there are values \(0 < z_2 (\mu) \leq z_2^+ (\mu)\) such that for \(0 < \mu < 1/2\)

- If \(z_2 \in (0, z_2^-) \cup (z_2^+, 1)\), then there are 4 elliptic relative equilibria.
- If \(z_2 \in (z_2^-, z_2^+)\), then there is not elliptic relative equilibrium.
- If \(z_2 \in (z_2^-, z_2^+)\), then there are 2 elliptic relative equilibria.
- If \(\mu = 1/2\), then there are 4 elliptic relative equilibria.

**Proof.** From Definition 2.1, we have

\[ \dot{q}_j = i \omega r_j e^{i(\omega t + \alpha_j)}, \quad \ddot{q}_j = -\omega^2 r_j e^{i(\omega t + \alpha_j)}, \]

\[ \dot{q}_j \odot \dot{q}_j = \omega^2 r_j^2, \quad q_1 \odot q_2 = r_1 r_2 \cos(\alpha_1 - \alpha_2) + \sigma z_1 z_2 := \alpha. \]
Let be $r = (1 - \alpha^2)^{3/2}$, then replacing relations (7) into the first equation of (3) we get

$$-\omega^2 r_1 e^{i(\omega t+\alpha_1)} = \left[-\sigma \omega^2 r_1^3 e^{i\alpha_1} + \frac{m_2}{r} \left( r_2 e^{i\alpha_2} - \sigma \alpha r_1 e^{i\alpha_1} \right) \right] e^{i\omega t}$$

$$- \left[ \sigma \omega^2 r_1^2 z_1 + \frac{m_2}{r} (\sigma \alpha z_1 - z_2) \right] e_3.$$  

We observe that in order to get a relative equilibrium we must have that the coefficient of $e_3$ vanish, then comparing the two first components we obtain

$$-\omega^2 r_1 = -\sigma \omega^2 r_1^3 + \frac{m_2}{r} \left( r_2 \cos(\alpha_2 - \alpha_1) - \sigma \alpha_1 \right) + i \frac{m_2}{r} \sin(\alpha_2 - \alpha_1).$$

from here we get that $\alpha_1 - \alpha_2 = k\pi, k = 0, 1$. So, it seems that two types of solutions do exist. With this condition on the angles $\alpha_1$ and $\alpha_2$, we obtain that $\alpha = (-1)^k r_1 r_2 + \sigma z_1 z_2$. Thus, from the real part in (9) and repeating this process with the second equation of (3) we obtain that the relative equilibria are given by the solutions of the following system of algebraic equations

$$\omega^2 r_1 z_1 = \frac{m_2}{r} (\sigma z_2 - \alpha_1 z_1),$$

$$\omega^2 r_2 z_2 = \frac{m_1}{r} (\sigma z_1 - \alpha_2 z_2).$$

By straightforward computations using (10), the normalized masses and the value of $\alpha$, we obtain that $z_1, z_2$ and $\mu$ are linked through the following equation

$$\frac{r_1 z_1}{r_2 z_2} = (-1)^{k+1} \left( \frac{1}{\mu} - 1 \right).$$

Note that $k = 0$ implies that $z_1 z_2 < 0$, which means that in $\mathbb{H}^2$ we must have necessarily $k = 1$. The same happens in $\mathbb{S}^2$, indeed, taking $k = 0$ and $\sigma = 1$ in the first equation of (10) we obtain

$$\omega^2 r_2 = \frac{m_2}{r} \left( \frac{z_2 r_1}{z_1} - r_2 \right) < 0.$$

Therefore, for both surfaces, we must have $k = 1$, i.e., the initial conditions of the particles has the form $P_1 = (r_1, 0, z_1)$ and $P_2 = (-r_2, 0, z_1)$. Even more, for $\mathbb{S}^2$, the particles must be located on the same hemisphere, thus, without loss of generality, we can consider $z_1, z_2 \in (0, 1)$.

The polynomial form of (11) is

$$p(z_1) = z_1^4 - z_1^2 + \beta^2 z_2^2 (1 - z_2^2) = 0, \text{ with } \beta = \frac{1}{\mu} - 1,$$

whose discriminant is given by

$$\Delta = 1 - 4\beta^2 z_2^2 (1 - z_2^2),$$

then, for each solution of (12), we have two relative equilibria with angular velocities $\pm \omega$, respectively. For the case $\sigma = -1$, we observe

$$p(1) = \beta z_2^2 (1 - z_2^2) < 0, \quad p'(z_1) = 2z_1^2 (2z_1^2 - 1) > 0 \text{ and } \Delta > 0, \quad \forall z_1, z_2 > 1,$$
which implies that there is a unique solution of (12). For $\sigma = 1$, there are two values of \( z_2 \), 0 < \( z_2^- \leq z_2^+ < 1 \), given by

\[
z_2^\pm = \sqrt{\frac{1 - \mu \pm \sqrt{1 - 2\mu}}{2(1 - \mu)}},
\]

such that:

- If \( z_2 \in (0, z_2^-) \cup (z_2^+, 1) \), then (11) has two solutions \( z_1^-, z_1^+ \in (0, 1) \) given by

\[
z_1^\pm = \sqrt{\frac{1 \pm \sqrt{\Delta}}{2}}.
\]

- If \( z_2 \in (z_2^-, z_2^+) \), then (11) has no solutions.
- If \( z_2 \in \{ z_2^-, z_2^+ \} \), then (11) has a unique solution \( z_1 = \frac{1}{\sqrt{2}} \).

For the special case of equal masses, we have two types of relative equilibria, namely, \( z_1 = z_2 \) (valid for \( \sigma = \pm 1 \)) and \( z_1 = \sqrt{1 - z_2^2} \) (valid only for \( \sigma = 1 \)). In \( \mathbb{H}^2 \), the relative equilibria are determined by \( z_1^+ \), which is defined for all \( z_2 > 1 \). □

**Remark 2.1.** For the case of \( \mathbb{S}^2 \), we obtain that \( z_1 = 0 \), if and only if, \( z_2 = 0 \), while for \( \mathbb{H}^2 \), \( z_1 = 1 \), if and only if, \( z_2 = 1 \). Both cases represent a singularity of equations (3).

### 3. The restricted three body problem on curved spaces

In this section we introduce a very particular kind of three body problem defined on spaces of constant curvature, the restricted three body problem, which consists in the study of the dynamics of a particle of infinitesimal mass moving on \( \mathbb{M}^2 \) under the influence of the attraction of two positive masses (called primaries) whose dynamics is known, in our case, they move on an elliptic relative equilibrium described in the previous section. The equations of motion for the infinitesimal mass with position \( \textbf{q} = (x_1, x_2, x_3) \) are given by

\[
\dot{\textbf{q}} = \sigma (\dot{\textbf{q}} \odot \dot{\textbf{q}}) \textbf{q} + \mu \frac{\textbf{q}_1(t) - \sigma (\textbf{q} \odot \textbf{q}_1(t)) \textbf{q}}{[\sigma - \sigma (\textbf{q} \odot \textbf{q}_1(t))^2]^{3/2}} + (1 - \mu) \frac{\textbf{q}_2(t) - \sigma (\textbf{q} \odot \textbf{q}_2(t)) \textbf{q}}{[\sigma - \sigma (\textbf{q} \odot \textbf{q}_2(t))^2]^{3/2}},
\]

with the corresponding restrictions

\[
\sigma (\textbf{q} \odot \textbf{q}) = 1, \quad \textbf{q} \odot \dot{\textbf{q}} = 0.
\]

In order to avoid the above restrictions (14), we introduce intrinsic coordinates by means of the stereographic projection (see [10] or [26] for more details)

\[
\Pi_\epsilon : \mathbb{M}^2 \sigma \rightarrow \mathbb{R}^2, \quad Q \mapsto q
\]
with \( Q = (x_1, x_2, x_3) \) and \( q = (x, y) \) such that

\[
x = \frac{x_1}{1 - \epsilon x_3}, \quad y = \frac{x_2}{1 - \epsilon x_3}.
\]

The inverse function is given by

\[
x_1 = \frac{2x}{\Delta}, \quad x_2 = \frac{2y}{\Delta}, \quad x_3 = \epsilon \frac{\Delta - 2}{\Delta},
\]

with \( \Delta = 1 + \sigma (x^2 + y^2) \). For \( \sigma = -1 \), we take \( \epsilon = -1 \); whereas for \( \sigma = 1 \), \( \epsilon = 1 \) if the projection is taken from the north pole and \( \epsilon = -1 \) if the projection is taken from the south pole. The metric of \( \mathbb{M}^2_\sigma \) becomes

\[
ds^2 = \lambda(x, y)(dx^2 + dy^2),
\]

where \( \lambda(x, y) = \frac{4}{\Delta^2} \) is the conformal factor. We denote by \( \mathcal{R}^2_{sph} \) the Riemann space \( (\mathbb{R}^2, g) \), where \( g \) is the metric

\[
g = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix},
\]

and we call it the curved plane. On \( \mathcal{R}^2_{sph} \), the kinetic energy takes the form

\[
T = \frac{1}{2} (\dot{q}, \dot{q})_g = \frac{2}{\Delta^2}(\dot{x}^2 + \dot{y}^2),
\]

while the potential in the coordinates \( (x, y) \) becomes

\[
U(x, y, t) = \frac{\mu \alpha_1}{\sqrt{\sigma(\Delta^2 - \alpha_1^2)}} + \frac{(1 - \mu)\alpha_2}{\sqrt{\sigma(\Delta^2 - \alpha_2^2)}},
\]

where

\[
\alpha_1 = 2\sigma r_1(x, y) \cdot (\cos \omega t, \sin \omega t) + \epsilon z_1(\Delta - 2),
\]

\[
\alpha_2 = -2\sigma r_2(x, y) \cdot (\cos \omega t, \sin \omega t) + \epsilon z_2(\Delta - 2).
\]

The Euler’s equation associated to the Lagrangian \( L = T + U \) gives the equations of motion

\[
\ddot{x} = \frac{4\dot{x}}{\Delta}(x\dot{x} + y\dot{y}) - \frac{2x}{\Delta}(\dot{x}^2 + \dot{y}^2) + \frac{\Delta^2}{4} U_x,
\]

\[
\ddot{y} = \frac{4\dot{y}}{\Delta}(x\dot{x} + y\dot{y}) - \frac{2y}{\Delta}(\dot{x}^2 + \dot{y}^2) + \frac{\Delta^2}{4} U_y.
\]

In addition, we can consider that the dynamics of the infinitesimal mass particle is governed by a Hamiltonian system \( (H, \mathcal{R}^2_{sph}, \Omega) \), with Hamiltonian \( H = T - U \). The symplectic two form is given by
\[ \Omega = dx \wedge dp_x + dy \wedge dp_y = d(x dp_x + y dp_y), \]

with
\[ p_x = \frac{4}{\Delta^2} \dot{x}, \quad p_y = \frac{4}{\Delta^2} \dot{y}. \] (17)

Thus, in terms of the conjugate variables \((p_x, p_y)\), the Hamiltonian function takes the form
\[ H(x, y, p_x, p_y, t) = \frac{\Delta^2}{8} (p_x^2 + p_y^2) - U(x, y, t). \] (18)

In order to eliminate the dependence of the time in the potential, we introduce rotating coordinates. If \(q = (x, y)\), \(p = (p_x, p_y)\), \(Q = (u, v)\) and \(P = (p_u, p_v)\), then the symplectic change of coordinates \((q, p) = (R_{ot} Q, R_{ot} P)\), where
\[ R_{ot} = \begin{pmatrix} \cos o t & -\sin o t \\ \sin o t & \cos o t \end{pmatrix}, \]
transforms (18) into an autonomous Hamiltonian \(H\) given by
\[ H = \frac{\Delta^2}{8} (p_u^2 + p_v^2) + \omega (vp_u - up_v) - U(u, v), \] (19)

where
\[ U(u, v) = \frac{\mu v_1}{\sqrt{\sigma (\Delta^2 - v_1^2)}} + \frac{(1 - \mu) v_2}{\sqrt{\sigma (\Delta^2 - v_2^2)}}, \]

and
\[ v_1 = 2\sigma r_1 u + \epsilon z_1 (\Delta - 2), \quad v_2 = -2\sigma r_2 u + \epsilon z_2 (\Delta - 2), \quad \Delta = 1 + \sigma (u^2 + v^2). \]

The corresponding Hamilton’s equations are
\[ \dot{u} = \frac{\partial H}{\partial p_u} = \frac{\Delta^2}{4} p_u + \omega v, \]
\[ \dot{v} = \frac{\partial H}{\partial p_v} = \frac{\Delta^2}{4} p_v - \omega u, \]
\[ \dot{p}_u = -\frac{\partial H}{\partial u} = -\frac{\sigma \Delta u}{2} (p_u^2 + p_v^2) + \omega p_v + U_u, \]
\[ \dot{p}_v = -\frac{\partial H}{\partial v} = -\frac{\sigma \Delta v}{2} (p_u^2 + p_v^2) - \omega p_u + U_v, \] (20)

with
\[ U_u = \frac{\partial U}{\partial u} = 2\Delta \left[ \mu \frac{-u v_1 + \Delta (r_1 + \epsilon z_1 \mu)}{l_1^{3/2}} + (1 - \mu) \frac{-u v_2 + \Delta (-r_2 + \epsilon z_2 \mu)}{l_2^{3/2}} \right], \]
\[ U_v = \frac{\partial U}{\partial v} = 2\omega \Delta \left[ \mu \frac{\epsilon z_1 \Delta - v_1}{l_1^{3/2}} + (1 - \mu) \frac{\epsilon z_2 \Delta - v_2}{l_2^{3/2}} \right], \] (21)

and \(l_k = \Delta^2 - v_k^2, \ k = 1, 2.\)
Remark 3.1. We observe that through the projection $\Pi_{-1}$, the primaries are located at $P_i = (u_i^*, 0)$, with

$$
 u_1^* = \frac{\sigma (1 - z_1)}{r_1}, \quad u_2^* = -\frac{\sigma (1 - z_2)}{r_2}.
$$

(22)

For the case of $S^2$ the antipodal singularities are at $\tilde{P}_i = (\tilde{u}_i^*, 0)$, with

$$
 \tilde{u}_1^* = -\frac{(1 + z_1)}{r_1}, \quad \tilde{u}_2^* = \frac{(1 + z_2)}{r_2}.
$$

(23)

Through the projection $\Pi_{+1}$, the values of $u^*$ and $\tilde{u}^*$ are exchanged.

4. Relative equilibria for the symmetric R3BP

In this section we consider the study of relative equilibria just for the symmetric case, that is, the primaries are rotating uniformly on opposite sides of the same fixed parallel of radius $r_1 = r_2 = a$. It is easy to check that the angular velocity of the primaries is given by $\omega^2 = (8a^3 (1 - \sigma a^2)^{3/2})^{-1}$ and they are located symmetrically on the $u$-axis.

All equilibria of system (20) have the form $((u, v), -\frac{4\omega u}{\Delta^2}, \frac{4\omega v}{\Delta^2})$. They are obtained by solving the system

$$
 F_1(u, v) = \frac{4\omega^2 u (2-\Delta)}{\Delta^3} + U_u(u, v) = 0,
$$

$$
 F_2(u, v) = \frac{4\omega^2 v (2-\Delta)}{\Delta^3} + U_v(u, v) = 0.
$$

(24)

The next result follows easily using (24).

Proposition 4.1. For $\sigma = 1$, the north and south poles are equilibrium solutions for any $0 < a < 1$, whereas for $\sigma = -1$, the bottom point (by short the pole) is an equilibrium for any $a > 0$.

Lemma 4.1. A necessary condition to have an equilibrium solution of (20) when $r_1 = r_2$, is that $u \cdot v = 0$.

Proof. Suppose that $u \cdot v \neq 0$, then from (24) we obtain that $v F_1 = u F_2$ or equivalently

$$
 0 = v F_1 - u F_2 = a v \Delta^2 \left( \frac{1}{l_1^{3/2}} - \frac{1}{l_2^{3/2}} \right),
$$

since $v \neq 0$, then

$$
 0 = l_2 - l_1 = 8 \epsilon au \sqrt{1 - \sigma a^2 (2 - \Delta)},
$$

and since $u \neq 0$, we must necessarily have $\Delta = 2$; replacing this value into (24) we obtain

$$
 F_1 = \frac{\epsilon \sqrt{1 - \sigma a^2} u}{(\sigma (1 - a^2 u^2))^{3/2}}, \quad F_2 = \frac{\epsilon \sqrt{1 - \sigma a^2} v}{(\sigma (1 - a^2 u^2))^{3/2}}.
$$
which contradicts the fact that \((u, v)\) satisfies (24). Therefore \(u \cdot v = 0\) and the proof is complete. \(\square\)

For the equilibrium points that are not located at the poles, we define the isosceles equilibria as the equilibria located on the \(v\)-axis and the collinear equilibria as the equilibria located on the \(u\)-axis.

Isosceles equilibria are given by the zeroes of

\[
 f_{is}(v; a) = \frac{\omega^2 (1 - \sigma v^2)}{(1 + \sigma v^2)^4} - \frac{\sqrt{1 - \sigma a^2}}{l^{3/2}},
\]

where \(l = \sigma \left( (1 + \sigma v^2)^2 - (1 - \sigma a^2) (1 - \sigma v^2)^2 \right)\), whereas collinear equilibria are given by the zeroes of

\[
 f_{col}(u; a) = \frac{4\omega^2 u (1 - \sigma u^2)}{(1 + \sigma u^2)^4} - \frac{\rho_1}{l_1^{3/2}} - \frac{\rho_2}{l_2^{3/2}},
\]

where

\[
 \rho_j = 2u \sqrt{1 - \sigma a^2} + (-1)^j a (1 - \sigma u^2),
\]

\[
 l_1 = \sigma \left( (1 + \sigma u^2)^2 - \left(2a\sigma u + \sqrt{1 - \sigma a^2} (1 - \sigma u^2) \right)^2 \right),
\]

\[
 l_2 = \sigma \left( (1 + \sigma u^2)^2 - \left(-2a\sigma u + \sqrt{1 - \sigma a^2} (1 - \sigma u^2) \right)^2 \right).
\]

**Remark 4.1.** We observe that in the case of \(\sigma = 1\), that is, on \(\mathbb{S}^2\), the equilibria cannot be in the equator (in this case we have antipodal singularities). We also observe that the equilibria are given by pairs \((\pm u, 0)\) and \((0, \pm v)\), so from here on, we will consider \(u, v > 0\) (of course here the poles have been excluded).

**Proposition 4.2.** For \(\sigma = 1\), all isosceles equilibria are symmetrically located on the upper hemisphere \(H^+\). There are three bifurcation values given by

\[
 a_2 \approx 0.396255849301291..., \quad a_3 \approx 0.797325392579823..., \quad a_4 = \frac{\sqrt{4 - \sqrt{2}}}{2}.
\]

Depending on the value of the parameter \(a \in (0, 1)\), the number of isosceles equilibria are given in Table 1.
\textbf{Proof.} From equation (24), the value \( v \) must satisfy the equation

\[
\frac{\omega^2(1-v^2)}{\Delta^4} = \frac{\sqrt{1-a^2}}{r^{3/2}} > 0,
\]  

(28)

with \( \Delta = 1 + v^2 \), which implies that \( |v| < 1 \), i.e., the equilibrium always remains on the upper hemisphere \( H^+ \) (recall the change of coordinates (15)). A solution of (28) is a curve \((a, v(a))\) in the \( av\)-plane. A first possible value of bifurcation for the parameter \( a \) appears when the equilibrium collides with the north pole, i.e., when \( v(a) = 0 \), and by replacing it into (28) we obtain the equation \( 8a^4 - 16a^2 + 7 = 0 \), whose unique solution in the interval \((0, 1)\) is given by

\[
a = \frac{1}{2} \sqrt{4 - \sqrt{2}}.
\]

Now, the solutions of (28) are contained in the set of solutions of the polynomial

\[
p(v) = 64a^{14} - 256a^{12} + 384a^{10} - 256a^8 + 63a^6 \\
+ (512a^{14} - 2048a^{12} + 3072a^{10} - 2048a^8 + 520a^6 - 12a^4) v^2 \\
+ (1792a^{14} - 7168a^{12} + 10752a^{10} - 7168a^8 + 1764a^6 + 72a^4 - 48a^2) v^4 \\
+ (3584a^{14} - 14336a^{12} + 21504a^{10} - 14336a^8 + 3640a^6 - 180a^4 + 192a^2 - 64) v^6 \\
+ (4480a^{14} - 17920a^{12} + 26880a^{10} - 17920a^8 + 4410a^6 + 240a^4 - 288a^2 + 128) v^8 \\
+ (3584a^{14} - 14336a^{12} + 21504a^{10} - 14336a^8 + 3640a^6 - 180a^4 + 192a^2 - 64) v^{10} \\
+ (1792a^{14} - 7168a^{12} + 10752a^{10} - 7168a^8 + 1764a^6 + 72a^4 - 48a^2) v^{12} \\
+ (512a^{14} - 2048a^{12} + 3072a^{10} - 2048a^8 + 520a^6 - 12a^4) v^{14} \\
+ (64a^{14} - 256a^{12} + 384a^{10} - 256a^8 + 63a^6) v^{16}.
\]

(29)

The number of roots of \( p(v) \) changes when it reaches a double zero, which occurs for the values of the parameter \( a \) such that \( q(a) = \text{Resultant}(p(v), p'(v)) = 0 \), where

\[
q(a) = \alpha(1-a^2)^7 a^7 (8a^4 - 16a^2 + 7)^3 (8a^4 - 16a^2 + 9)^3 \\
\times (16384a^{16} - 81920a^{14} + 163840a^{12} - 163840a^{10} + 81920a^8 - 16384a^6 + 27)^4,
\]

(30)

which has three zeroes in the interval \((0, 1)\) given by

\[
a_2 \approx 0.3962558493012911, \quad a_3 \approx 0.797325392579823, \quad a_4 = \frac{1}{2} \sqrt{4 - \sqrt{2}}.
\]

After taking a test value of the parameter on each interval determined by \( a_2, a_3, a_4 \), we obtain:

- If \( a \in (0, a_2) \), then there exist two pairs of symmetric isosceles equilibria, namely, \( \pm L_{v_1}, \pm L_{v_2} \);
- If \( a = a_2 \), then there is one pair of symmetric isosceles equilibria, namely, \( \pm L_{v_12} \);
- If \( a \in (a_2, a_3) \), then there are no isosceles equilibria;
- If \( a = a_3 \), then there is one pair of symmetric isosceles equilibria, namely, \( \pm L_{v_{34}} \).
FIG. 1. Parameter bifurcation diagram for the existence of equilibria in the symmetric R3BP on $S^2$: (a) Isosceles equilibria with $a \in (0, a_2]$; (b) Isosceles equilibria with $a \in [a_3, 1)$; (c) Collinear equilibria with $a \in (0, a_1]$; (d) Collinear equilibria with $a \in [a_5, 1)$.

- If $a \in (a_3, a_4)$, then there are two pairs of isosceles symmetric equilibria, namely, $\pm L_{v3}, \pm L_{v4}$;
- If $a \in [a_4, 1)$, then there is one pair of symmetric isosceles equilibria, namely, $\pm L_{v5}$.

Thus, our analysis agrees with Table 1 and the relative position can be viewed in Fig. 1 (a) and (b).

**Proposition 4.3.** For $\sigma = -1$ and any value of the parameter $a > 0$, there are exactly two isosceles equilibria symmetrically located.

**Proof.** From equation (25), the isosceles equilibria for $\sigma = -1$ are obtained by solving $g_1(v) = g_2(v)$ where

$$g_1(v) = \frac{\tilde{\omega}(v^2 + 1)}{(1 - v^2)^{\frac{3}{2}}}, \quad g_2(v) = \frac{1}{l^{3/2}},$$

with $\tilde{\omega} = (8a^3(a^2 + 1)^2)^{-1}$. The functions $g_1$ and $g_2$ have the following properties:

$$g_1'(v) = \frac{2\tilde{\omega}v(3v^2 + 5)}{(1 - v^2)^{\frac{5}{2}}} > 0, \quad \text{and} \quad g_2'(v) = -\frac{6v(a^2v^2 + a^2 + 2)}{l^{5/2}} < 0,$$
Table 2

| Number of collinear relative equilibria on $S^2$ for different values of the parameter $a$. |
|----------------------------------|------------------|------------------|------------------|
| $H^+$  | $(0, a_1)$ | $a_1$ | $(a_1, a_5]$ | $(a_5, 1)$ |
| $H^-$  | 0 | 0 | 0 | 2 |

$g_1(0) = \tilde{\omega} < \frac{1}{a^3} = g_2(0)$, \quad $g_1(v) \to +\infty$, when $v \to 1$,

which assure that $g_1$ and $g_2$ intersect at a single point in the interval $0 < v < 1$ for any value of $a > 0$. □

**Proposition 4.4.** For $\sigma = 1$, the collinear relative equilibria are symmetrically located. There are two bifurcation values given by

$$a_1 \approx 0.337849954103973..., \quad a_5 = \frac{\sqrt{3}}{2} \approx 0.8660254...$$

The number of relative equilibria in the intervals determined by the bifurcation values are given in Table 2.

**Proof.** We define

$$\alpha = \frac{4\omega^2 u (1 - u^2)}{(u^2 + 1)^4},$$

$$\beta = \frac{a (1 - u^2) - 2\sqrt{1 - a^2}u}{l_1^{3/2}} - \frac{a (1 - u^2) + 2\sqrt{1 - a^2}u}{l_2^{3/2}},$$

with $l_1$ and $l_2$ as defined in (27) with $\sigma = 1$. Then, solving $f_{\text{col}}(u; a) = 0$ is equivalent to solve the equation

$$\alpha + \beta = 0. \quad (31)$$

In order to eliminate the roots in the denominators of $\beta$, we need to solve the equation

$$\alpha^2 - \beta^2 = 0. \quad (32)$$

In developing $\beta^2$, the term $(l_1 \cdot l_2)^{3/2}$ appears in the denominator and it is given by

$$(l_1 \cdot l_2)^{3/2} = \begin{cases} (a - 2u + u^2)^3(a + 2u + u^2)^3 & \text{if } u \in (0, u_1^*) \cup (\tilde{u}_2^*, +\infty), \\ -(a - 2u + u^2)^3(a + 2u + u^2)^3 & \text{if } u \in (u_1^*, \tilde{u}_2^*). \end{cases} \quad (33)$$

For $u \in (0, u_1^*) \cup (\tilde{u}_2^*, +\infty)$, equation (32) becomes

$$\frac{1}{4a^6 (1 - a^2)^3 (u^2 + 1)^8} - \frac{64a^2 (1 - a^2)}{(au^2 + a - 2u)^4 (au^2 + a + 2u)^4} = 0,$$
or equivalently, in its polynomial form \( p_1(u) p_2(u) p_3(u) = 0 \), with

\[
\begin{align*}
p_1(u) &= a^2 (4a^2 - 5) u^4 + 2 (4a^4 - 5a^2 + 2) u^2 + a^2 (4a^2 - 5) , \\
p_2(u) &= a^2 (4a^2 - 3) u^4 + 2 (4a^4 - 3a^2 - 2) u^2 + a^2 (4a^2 - 3) , \\
p_3(u) &= a^4 (16a^4 - 32a^2 + 17) u^8 + 4a^2 (16a^6 - 32a^4 + 17a^2 - 2) u^6 \\
&\quad + 2 (48a^8 - 96a^6 + 51a^4 - 8a^2 + 8) u^4 + 4a^2 (16a^6 - 32a^4 + 17a^2 - 2) u^2 \\
&\quad + a^2 (16a^4 - 32a^2 + 17) . \tag{34}
\end{align*}
\]

We state that the polynomial \( p_3 \) does not provide solutions of (32), in fact, we obtain that the resultant between \( p_3(u) \) and \( p'_3(u) \) is given by

\[
\text{Resultant}(p_3(u), p'_3(u)) = \alpha a^{28} (1 - a^2)^{12} (16a^4 + 1)^2 (16a^4 - 32a^2 + 17)^3 \neq 0,
\]

for all \( a \in (0, 1) \) and \( \alpha \) a nonzero constant, which implies that the number of zeroes of \( p_3(u) \) does not vary for any value of \( a \), thus, we can put \( a = 1/2 \) and \( p_3 \) takes the form \( p_3(u) = \frac{1}{8} (5u^8 + 4u^6 + 126u^4 + 3u^2 + 5) \), which does not have zeroes which proves the statement. The zeroes of \( p_1 \) and \( p_2 \) can be easily computed and it is verified that the zeroes of \( p_1(u) \) do not provide zeroes of (32). Nevertheless, the polynomial \( p_2 \) gives one positive solution of (32)

\[
u_3 = \frac{1}{a} \sqrt{-4a^4 + 3a^2 + 2 + 2\sqrt{-4a^4 + 3a^2 + 1}} \\over{4a^2 - 3} , \tag{35}
\]

which is defined for \( a \in (a_5, 1) \), with \( a_5 = \frac{1}{2} \). It is easy to verify that (35) satisfies \( u > 1 \), and therefore, the corresponding collinear equilibrium is always located at the lower hemisphere.

If \( u \in (u_1^+, u_2^+) \), then equation (32) assumes the form

\[
\frac{u^2 (1 - u^2)^2}{4a^6 (1 - a^2)^3 (u^2 + 1)^8} - \frac{4 (a^2 u^4 - 6a^2 u^2 + a^2 + 4u^2)^2}{(au^2 + a - 2u)^3 (au^2 + a + 2u)^3} = 0,
\]

whose polynomial form is given by

\[
\begin{align*}
p_4(u) &= (16a^{16} - 48a^{14} + 48a^{12} - 16a^{10}) u^{24} \\
&\quad + (-64a^{16} + 320a^{14} - 576a^{12} + 448a^{10} - 127a^8) u^{22} \\
&\quad + (-480a^{16} + 1696a^{14} - 1952a^{12} + 480a^{10} + 518a^8 - 272a^6) u^{20} \\
&\quad + (192a^{16} - 3008a^{14} + 9920a^{12} - 13632a^{10} + 8589a^8 - 2112a^6 + 96a^4) u^{18} \\
&\quad + a_{16} u^{16} + a_{14} u^{14} + a_{10} u^{10} + a_8 u^8 \\
&\quad + (192a^{16} - 3008a^{14} + 9920a^{12} - 13632a^{10} + 8589a^8 - 2112a^6 + 96a^4) u^6 \\
&\quad + (-480a^{16} + 1696a^{14} - 1952a^{12} + 480a^{10} + 518a^8 - 272a^6) u^4 \\
&\quad + (-64a^{16} + 320a^{14} - 576a^{12} + 448a^{10} - 127a^8) u^2 \\
&\quad + 16a^{16} - 48a^{14} + 48a^{12} - 16a^{10} , \tag{36}
\end{align*}
\]

with
\[ a_{16} = 5872a^{16} - 30928a^{14} + 64720a^{12} - 67312a^{10} + 34824a^8 - 7232a^6 + 192a^4 - 256a^2, \]
\[ a_{14} = 16256a^{16} - 79232a^{14} + 154496a^{12} - 150656a^{10} + 73458a^8 - 14272a^6 - 96a^4 + 256, \]
\[ a_{12} = 21952a^{16} - 105280a^{14} + 202048a^{12} - 193984a^{10} + 93156a^8 - 17760a^6 - 384a^4 + 512a^2 - 512, \]
\[ a_{10} = 16256a^{16} - 79232a^{14} + 154496a^{12} - 150656a^{10} + 73458a^8 - 14272a^6 - 96a^4 + 256, \]
\[ a_{8} = 5872a^{16} - 30928a^{14} + 64720a^{12} - 67312a^{10} + 34824a^8 - 7232a^6 + 192a^4 - 256a^2. \]

The resultant between \( p_4(u) \) and \( p_4'(u) \) is given by

\[ \text{Resultant}(p_4(u), p_4'(u)) = \alpha \tilde{p}(a), \quad (37) \]

where \( \alpha \) is a nonzero constant and

\[ \tilde{p}(a) = (16777216a^{24} - 100663296a^{22} + 264241152a^{20} - 398458880a^{18} + 380436480a^{16} - 238288896a^{14} + 98435072a^{12} - 27525120a^{10} + 6138624a^8 - 1136128a^6 + 54336a^4 - 10560a^2 + 1511)^4. \]

The zeroes of \( \tilde{p}(a) \) are

\[ a_1 \approx 0.33784995410397306 \quad \text{and} \quad \tilde{a} \approx 0.941199983272388. \]

By considering a test value of the parameter on each interval determined by \( a_1, a_5 \) and \( \tilde{a} \), it verifies that

- If \( a \in (0, a_1) \), there exist two pairs of symmetric collinear equilibria, namely, \( \pm L_{u_1}, \pm L_{u_2} \);
- If \( a = a_1 \), then there is one pair of symmetric collinear equilibria, namely, \( \pm L_{u_{12}} \);
- If \( a \in (a_1, a_5) \), then there are no collinear equilibria;
- If \( a \in (a_5, \tilde{a}) \), then there is one pair of symmetric equilibria, which is the one determined by (35), namely, \( \pm L_{u_3} \);
- If \( a \in [\tilde{a}, 1) \), then there is one pair of symmetric equilibria, which is the one determined by (35), namely, \( \pm L_{u_3} \).

From the above, we conclude that \( a = \tilde{a} \) does not provide a bifurcation value with respect to the number of relative equilibria and therefore, the number of collinear relative equilibria changes at \( a = a_1 \) and \( a = a_5 \) (see Fig. 1(c) and (d)), which completes the proof. \( \Box \)

**Proposition 4.5.** For \( \sigma = -1 \) and any value of the parameter \( a > 0 \), there are exactly two collinear equilibria symmetrically located.

**Proof.** From equation (26) with \( \sigma = -1 \), we have that the relative equilibria are given by the intersection of two functions \( f_1 \) and \( f_2 \), with
\[ f_1(u) = \frac{4\omega^2 u(1 + u^2)}{(1 - u^2)^4}, \quad f_2(u) = \frac{\rho_1}{l_1^{3/2}} + \frac{\rho_2}{l_2^{3/2}}. \]

The function \( f_1 \) is increasing, positive and \( f_1(u) \to +\infty \) when \( u \to 1 \). We separate our analysis in two intervals, \((0, u^*_1)\) and \((u^*_1, 1)\). For the first one, it is easy to verify that \( \rho_1 < 0 \) and \( \rho_2 > 0 \), which implies that \( f_2 := -\rho_1/l_1^{3/2} + \rho_2/l_1^{3/2} > 0 \). Since \( sgn(f_2) = sgn(f_2 \cdot \bar{f}_2) \), where

\[ f_2 \cdot \bar{f}_2 = -\frac{16a\sqrt{1 + a^2}u(1 + u^2)(4u^2 + a^2(1 + 6u^2 + u^4))}{(-4u^2 + a^2(1 - u^2)^2)^4} < 0, \]

it follows that \( f_2(u) < 0 \) for all \( u \in (0, u^*_1) \) and therefore, there are no equilibria in this interval. For \( u \in (u^*_1, 1) \), we have \( \rho_1, \rho_2 > 0 \), then \( f_2(u) > 0 \) and \( f_2(u) \to +\infty \) when \( u \to u^*_1 \), which implies that the graphics \( f_1 \) and \( f_2 \) must intersect at least at one point. Now, we compute

\[ f'_2(u) = \frac{\beta_1 - \beta_2}{l_1^{5/2}} - \frac{\beta_1 + \beta_2}{l_2^{5/2}}, \]

where

\[ \beta_1 = 4au^2 \left( a^2u^4 + (10a^2 + 8)u^2 + 5a^2 + 4 \right), \quad \beta_2 = 4\sqrt{a^2 + 1} \left( 5a^2u^4 + (10a^2 + 4)u^2 + a^2 \right). \]

It is clear that \( \beta_1 + \beta_2 > 0 \) and \( sgn(\beta_1 - \beta_2) = sgn(\beta_1^2 - \beta_2^2) \). Since

\[ \beta_1^2 - \beta_2^2 = -16 \left( -au^2 + a - 2u \right)^2 \left( -au^2 + a + 2u \right)^2 \left( a^2 + 1 - a^2u^2 \right) < 0, \]

it follows that \( f'_2(u) < 0 \). Therefore, the graphics of \( f_1 \) and \( f_2 \) intersect at a single point located in the interval \((u^*_1, 1)\), which completes the proof. \( \square \)

5. Linear stability of the relative equilibria for the symmetric R3BP

Since we have only one parameter \( a \), it is possible to carry out a complete study of the linear stability of each equilibrium point on both surfaces. Moreover, a new bifurcation value appears for a certain equilibrium on \( S^2 \).

The linearization matrix at an equilibrium solution \( \left( u, v, -\frac{4uv}{\Delta^2}, \frac{4uv}{\Delta^2} \right) \), with \( uv = 0 \), has the form

\[ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \]

where

\[ A_1 = \begin{pmatrix} 0 & \omega - \frac{4v^2\sigma_0}{\Delta} \\ \frac{4u^2\sigma_0}{\Delta} - \omega & 0 \end{pmatrix}, \quad A_2 = \frac{\Delta^2}{4} I_2. \]
A_3 = Hess(U) - \frac{8 \omega^2 (u^2 + v^2)}{\Delta^4} \begin{pmatrix} \sigma \Delta + 2u^2 & 0 \\ 0 & \sigma \Delta + 2v^2 \end{pmatrix}, \quad A_4 = -A_1^T.

The characteristic polynomial is given by the bi-quadratic polynomial

\[ p(\lambda) = \frac{\Delta^4}{16} \det B = \beta_4 \lambda^4 + \beta_2 \lambda^2 + \beta_0, \quad (38) \]

where

\[ B = A_3 + \frac{4}{\Delta^2} (A_1^T + \lambda I_2)(A_1 - \lambda I_2) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \quad (39) \]

It is easy to verify that \( U_{uv} = U_{vu} = 0 \) for \( uv = 0 \), so the entries \( b_{ij} \) are given by

\[ b_{11} = b_1 - \frac{4 \lambda^2}{\Delta^2}, \quad b_{12} = \frac{8 \omega (2 - \Delta) \lambda}{\Delta^3}, \quad b_{21} = -b_{12}, \quad b_{22} = b_2 - \frac{4 \lambda^2}{\Delta^2}, \]

with

\[ b_1 = \frac{4 \omega^2}{\Delta^3} \left( (\delta - 4 \sigma u^2)^2 - 2 (u^2 + v^2) (\delta \sigma + 2u^2) \right) - U_{uu}, \]

\[ b_2 = \frac{4 \omega^2}{\Delta^3} \left( (\delta - 4 \sigma v^2)^2 - 2 (u^2 + v^2) (\delta \sigma + 2v^2) \right) - U_{vv}. \quad (40) \]

In this way, the characteristic polynomial takes the form

\[ p(\lambda) = \lambda^4 + \beta_2 \lambda^2 + \beta_0, \]

where

\[ \beta_2 = \frac{4 \omega^2 (2 - \Delta)^2}{\Delta^2} - \frac{\Delta^2}{4} (b_1 + b_2) = \frac{2 \omega^2 (1 + u^4 + v^4)}{\Delta^2} - \frac{1}{4} \Delta^2 (U_{uu} + U_{vv}), \]

\[ \beta_0 = \frac{\Delta^4}{16} b_1 b_2 = \frac{\Delta^4}{16} \left( \frac{4 \omega^2 (3u^4 - v^4 - 8uv^2 + 1)}{\Delta^4} + U_{uu} \right) \left( \frac{4 \omega^2 (3v^4 - u^4 - 8uv^2 + 1)}{\Delta^4} + U_{vv} \right). \quad (41) \]

5.1. Linear stability for the isosceles relative equilibria

For \( u = 0 \), \( \beta_2 \) takes the form

\[ \beta_2 = \frac{(1 + \sigma v^2)^2 \sqrt{1 - \sigma a^2} (1 - \sigma v^2)}{l^{3/2}} + \frac{2 \omega^2 (v^4 + 1)}{(\sigma v^2 + 1)^2}, \]

and using the condition \( f_{is}(v; a) = 0 \), it follows that \( \beta_2 = \omega^2 \) and the characteristic polynomial becomes

\[ p(\lambda) = \lambda^4 + \omega^2 \lambda^2 + \beta_0. \]
Similarly, we obtain
\[ \beta_0 = \frac{12a^2v^2\omega^4 (1 - \sigma v^2) q(v; a)}{\Delta^2 l^2}, \]
with \( q(v; a) = (3 - 4\sigma a^2) v^4 + (8a^2 - 10\sigma) v^2 + 3 - 4\sigma a^2 \). The discriminant of \( p(\lambda) \) is given by
\[ \delta = \omega^4 \left( 1 - \frac{48a^2v^2(1 - \sigma v^2) q(v; a)}{\Delta^2 l^2} \right). \]

**Theorem 5.1.** For \( \sigma = 1 \), there exists \( a_0 \approx 0.36806586863987895 \ldots \) such that the stability of the isosceles relative equilibria is characterized as follows:

- If \( a \in (0, a_0) \), then \( \pm L_{v_1} \) are unstable (saddle–saddle) and \( \pm L_{v_2} \) are unstable (center-saddle);
- If \( a = a_0 \), then \( L_{v_{12}} \) is spectrally stable;
- If \( a \in (a_0, a_2) \), then \( \pm L_{v_1} \) are linearly unstable (saddle–saddle) and \( \pm L_{v_2} \) are unstable (center-saddle);
- If \( a = a_3 \), then \( L_{v_{34}} \) is spectrally stable;
- If \( a \in (a_3, a_4) \), then \( \pm L_{v_3} \) are linearly stable (center–center) and \( \pm L_{v_4} \) are unstable (center-saddle);
- If \( a \in [a_4, 1) \), then \( \pm L_{v_5} \) are unstable (center-saddle).

**Proof.** Letting \( x = \lambda^2 \), the characteristic polynomial can be written as
\[ \tilde{p}(x) = x^2 + \omega^2 x + \beta_0. \] (42)
If \( \beta_0 < 0 \), then the discriminant \( \delta = \omega^4 - 4\beta_0 > 0 \), which implies that both roots of (42), \( x_1 \) and \( x_2 \) are real and also, by Descartes’ rule of signs, it verifies that \( x_1 < 0 < x_2 \). So in this case, it follows that all isosceles relative equilibria with \( \beta_0 < 0 \), have two real and two pure imaginary eigenvalues and, therefore, the corresponding relative equilibria are unstable.

If \( \beta_0 = 0 \), then equation (42) has roots \( x_1 = 0 \) and \( x_2 = -\omega^2 \), which implies that all isosceles relative equilibria with \( \beta_0 = 0 \) have a double zero eigenvalue and two pure imaginary eigenvalues \( \lambda = \pm i\omega \), therefore, in this case the relative equilibria are spectrally stable. Moreover, the system of algebraic equations \( f_1(v; a) = 0 \) and \( \beta_0(v; a) = 0 \) can be solved. In fact, its zeroes are given by solving the polynomial equation \( q(v; a) = 0 \), which has a unique solution such that \( v_0(a) < 1 \) and it is given by
\[ v_0(a) = \frac{4a^2 - 5 + 4\sqrt{1 - a^2}}{4a^2 - 3}, \] (43)
defined for \( a \in (0, a_5) \). Replacing (43) into (28) yields the polynomial equation
\[ 16384a^{16} - 81920a^{14} + 163840a^{12} - 163840a^{10} + 81920a^8 - 16384a^6 + 27 = 0, \]
which has two solutions into the interval \((0, 1)\) given by
where \( a_2 \) and \( a_3 \) are the bifurcation values given in Proposition 4.2.

For \( \beta_0 > 0 \), the solutions of (42) are either both complex (if \( \delta < 0 \)) or both negative real numbers (if \( \delta \geq 0 \)). The equation \( \delta = 0 \) is equivalent to the equation \( \delta = 0 \), where

\[
\delta(v) = -a^4v^{12} + (136a^2 - 190a^4)v^{10} + (769a^4 - 768a^2 - 16)v^8 + (-1156a^4 + 1264a^2 - 32)v^6 + (769a^4 - 768a^2 - 16)v^4 + (136a^2 - 190a^4)v^2 - a^4.
\]

while, solving \( f_{is}(v; a) = 0 \) is equivalent to solve (29). The equation

\[
\text{Resultant}(\delta(v), p(v)) = p_2(a) = 0,
\]  

where

\[
p_2(a) = \alpha a^{48}(1 - a^2)^2 \left(21734271936a^{40} - 282662535168a^{38} + 1695975211008a^{36} - 6218575773696a^{34} + 15546070486976a^{32} - 27980269380608a^{30} + 37298163546624a^{28} - 37280430091264a^{26} + 27937095457024a^{24} - 15500373869568a^{22} + 6189409662464a^{20} - 1686738185216a^{18} + 284917230288a^{16} - 26400323264a^{14} + 2957123840a^{12} - 1081023168a^{10} + 213652768a^8 - 133402224a^6 - 974976a^4 - 188480a^2 + 41975\right)^4,
\]

where \( \alpha \) is a nonzero constant. The polynomial \( p_2(a) \) gives the values of \( a \) where the curve \( \gamma = (a, v(a)) \), which satisfies \( f_{is}(v(a); a) = 0 \), crosses the boundary of the region of points \( (a, v) \) such that \( \delta(a, v) < 0 \). It is not difficult to verify that equation (45) provides only one solution for the parameter \( a \) in the corresponding interval, which is given by \( a_0 \approx 0.36806586863987895 \).

For a solution \( (a, v(a)) \) satisfying (28), we denote by \( \gamma_j = (a, v_j(a)) \), \( j = 1, 2, 3, 4, 5 \), the curves of equilibria in the plane \( (a, v) \), where \( L_{v_j} = (0, v_j(a)) \), is defined in its corresponding domain given by Proposition 4.2.

For \( a \in (0, a_2) \), we consider the two equilibria \( L_{v_1} \) and \( L_{v_2} \), with \( v_1(a) < v(a) < v_2(a) \). It is verified that \( \gamma_2 \) is always contained in the region \( \beta_0 < 0 \) (region I in Fig. 2) and by the analysis done above, the equilibrium \( L_{v_2} \) is always unstable, even more, it has one pair of opposite pure imaginary eigenvalues and two opposite nonzero real eigenvalues, so it is an unstable center-saddle. Nevertheless, when \( a = a_2 \) (see Proposition 4.2) we have that along the curve \( f_{is}(v; a) = 0 \), we have \( \beta_0 = 0 \), therefore, the equilibrium \( L_{v_{12}} \) has one pair of pure imaginary and one double zero eigenvalues, which implies the spectral stability of \( L_{v_{12}} \) (spectral stability means that all eigenvalues \( \lambda \) satisfy \( \lambda^2 \leq 0 \)).

For the equilibrium \( L_{v_1} \), the curve \( \gamma_1 \) is always contained in the region \( \beta_0 > 0 \) (region II in Fig. 2). If \( a \in (0, a_0) \), then \( \gamma_1 \) is contained in the region such that the discriminant \( \delta < 0 \), which implies that all its eigenvalues are complex with nonzero real part, i.e., \( L_{v_1} \) is an unstable equilibrium (saddle–saddle). If \( a \in (a_0, a_2) \), the curve \( \gamma_1 \) is contained in the region \( \delta > 0 \), which implies that all its eigenvalues are pure imaginary and different from each other, i.e., \( L_{v_1} \) is a linearly stable center–center. If \( a = a_0 \), then along the curve \( \gamma_1 \), we have \( \delta = 0 \), which implies that \( L_{v_1} \) has a double pure imaginary eigenvalue (i.e., 1 : 1 resonance), therefore, it is spectrally stable.
For $a = a_3$, along the curve $\gamma_4$ we have $\beta_0 = 0$, therefore the equilibrium $L_{v34}$ has one pair of pure imaginary and double zero eigenvalues, which gives the spectral stability of $L_{v34}$.

For $a \in (a_3, a_4)$, there are two equilibria $L_{v3} = (0, v_3(a))$ and $L_{v4} = (0, v_4(a))$, with $v_3(a) \leq v(a_3) \leq v_4(a)$. The curve $\gamma_3$ is always contained in the region $\beta_0 > 0, \delta > 0$ (corresponding to the region III in Fig. 2 (c)), this is because the only value of $a$, for which the curve $f_{is}(v; a) = 0$ crosses the curve $\delta = 0$ is $a = a_0$, which, by virtue of the above analysis, implies that the eigenvalues are all pure imaginary and different from each other, i.e., $L_{v3}$ is a linearly stable center–center. The curve $\gamma_4$ is contained in the region $\beta_0 < 0$ (i.e., region I in Fig. 2 (c)), which implies that the equilibrium $L_{v4}$ has one pair of opposite pure imaginary eigenvalues and two
opposite nonzero real eigenvalues, so it is an unstable center-saddle. The same occurs with the curve $γ_5$ when $a \in (a_4, 1)$ (see Fig. 2), thus, $L_{v5}$ is an unstable center-saddle. This finishes the proof of Theorem 5.1. \hfill \Box

For the case of negative curvature, we have the following result.

**Theorem 5.2.** For $\sigma = -1$, all isosceles equilibria are complex saddle, i.e., all of their eigenvalues are complex with nonzero real part.

**Proof.** Since $q(v; a) > 0$ for $\sigma = -1$, it follows that the eigenvalues are either complex (with nonzero real part) or pure imaginary. Using the same method as in the previous cases, we reduce $\frac{δ}{σ^2}$ and $f_{is}(v; a)$ to their polynomial form, namely, $p_1(v)$ and $p_2(v)$, respectively, and computing the resultant between them, we obtain a polynomial $p(a) = \text{Resultant}(p_1(v), p_2(v))$, which has a unique zero $a \approx 0.360458658$... and it generates one intersection between $p_1$ and $p_2$, but it is outside the domain $0 < v < 1$, therefore, the curve of equilibria does not cross the curve $δ = 0$, and even more, it is contained within the region $δ < 0$, thus, any isosceles equilibrium is a complex saddle. This finishes the proof of Theorem 5.2. \hfill \Box

5.2. Linear stability for the collinear relative equilibria

For $v = 0$, $β_2$ takes the form

$$β_2 = g_0 + \frac{Δ^2}{2} \left( \frac{g_1}{l_{5/2}^1} + \frac{g_2}{l_{5/2}^2} \right),$$

where $g_0 = \frac{2ω^2(1+u^4)}{Δ^2}$, $g_1 = g_{11} + g_{12}$, $g_2 = g_{11} - g_{12}$,

$$g_{11} = \sqrt{1 - a^2σ} \left( 1 - σu^2 \right) \left( a^2u^4 + (4 - 14a^2σ)u^2 + a^2 \right),$$

$$g_{12} = 2au \left( 3σa^2 - 2 + (8σ - 10a^2)u^2 + (3a^2σ - 2)u^4 \right),$$

and

$$b_1 = \frac{4ω^2}{Δ^2} (1 - 8σu^2 + 3u^4) + \left( \frac{h_1}{l_{3/2}^1} + \frac{h_2}{l_{3/2}^2} \right),$$

$$b_2 = \frac{4ω^2(1-σu^2)}{Δ^2} - 2Δ \left( \frac{k_1}{l_{3/2}^1} + \frac{k_2}{l_{3/2}^2} \right),$$

where $k_j = \sqrt{1 - a^2σ} - (-1)^j aσu$ and $h_j = h_{11} - (-1)^j h_{12}$, with

$$h_{11} = 4\sqrt{1 - a^2σ} \left( 2a^2σu^6 + 5a^2u^4 + (4 - 12a^2σ)u^2 + a^2 \right),$$

$$h_{12} = 2au \left( a^2u^6 + (4 - 3a^2σ)u^4 + (20σ - 25a^2)u^2 + 11σa^2 - 8 \right).$$

**Lemma 5.1.** Let $(u, 0)$, with $u > 0$ a collinear equilibrium. For $σ = 1$, $b_2 < 0$ if $0 < u < 1$ and $b_2 > 0$ if $u > 1$. For $σ = -1$, $b_2 < 0$. 
Proof. From condition \( f_{col}(u; a) = 0 \), we obtain that

\[
ub_2 = \frac{4\omega^2 u(1 - \sigma u^2)}{\Delta^3} - 2\Delta u \left( \frac{k_1}{l_1^{3/2}} + \frac{k_2}{l_2^{3/2}} \right) \\
= \Delta \left[ \frac{\rho_1}{l_1^{3/3}} + \frac{\rho_1}{l_1^{3/3}} - 2u \left( \frac{k_1}{l_1^{3/2}} + \frac{k_2}{l_2^{3/2}} \right) \right] \\
= \Delta \left[ \frac{\rho_1 - 2uk_1}{l_1^{3/2}} + \frac{\rho_2 - 2uk_2}{l_2^{3/2}} \right] \\
= a\Delta \left[ \frac{1}{l_2^{3/2}} - \frac{1}{l_1^{3/2}} \right],
\]

and the result follows from the fact that \( l_1 < l_2 \) for \( 0 < u < 1 \) and \( l_1 > l_2 \) if \( u > 1 \) and \( \sigma = 1 \). \( \square \)

Theorem 5.3. For \( \sigma = 1 \), the stability of the collinear equilibria is characterized as follows:

- If \( a \in (0, a_1) \), then \( L_{u_1} \) is unstable (center-saddle) and \( L_{u_2} \) is linearly stable.
- If \( a = a_1 \), then \( L_{u_{12}} \) is spectrally stable.
- If \( a \in (a_5, 1) \), then \( L_{u_3} \) is linearly stable.

Proof. First, we analyze the sign of \( \beta_2 \), which is equivalent to study the sign of

\[
4\beta_2/\Delta^2 = \tilde{g}_0 - \left( \frac{g_1}{l_1^{3/2}} + \frac{g_2}{l_2^{3/2}} \right),
\]

with \( \tilde{g}_0 = 8\omega^2(1 + u^4)/\Delta^4 \). Now we compute

\[
\tilde{g}_0 \left( \frac{g_1}{l_1^{3/2}} + \frac{g_2}{l_2^{3/2}} \right)^2 = \tilde{g}_0 \left( \frac{g_2^2 t_1^{5/2} + g_1^2 t_2^{5/2} + 2g_1 g_2}{l_1^{10} l_2^{10}} \right), \quad (47)
\]

where \( \tilde{t}_1 = au^2 - 2u + a, \tilde{t}_2 = au^2 + 2u + a \), the sign + is for \( u \in (0, u_1^*) \cup (\tilde{u}_2^*, +\infty) \) and the sign − for \( u \in (u_1^*, \tilde{u}_2^*) \). In the first case, (47) can be reduced to its polynomial form

\[
\beta_2^+(u) = r(u)(r(u) + s(u)),
\]

with

\[
r(u) = \left( 4a^{10} - 8a^8 + 3a^6 \right) u^{16} + \left( 48a^8 - 102a^6 + 60a^4 \right) u^{14} + \left( -80a^{10} + 320a^8 - 416a^6 + 208a^4 - 48a^2 \right) u^{12} + \left( -256a^{10} + 720a^8 - 698a^6 + 292a^4 - 96a^2 + 64 \right) u^{10}
\]
+ \left( -360a^{10} + 912a^8 - 774a^6 + 288a^4 - 96a^2 \right) u^8 \\
+ \left( -256a^{10} + 720a^8 - 698a^6 + 292a^4 - 96a^2 + 64 \right) u^6 \\
+ \left( -80a^{10} + 320a^8 - 416a^6 + 208a^4 - 48a^2 \right) u^4 \\
+ \left( 48a^8 - 102a^6 + 60a^4 \right) u^2 + 4a^{10} - 8a^8 + 3a^6, \\
\therefore s(u) = 2 \left( 1 + u^4 \right) \left( au^2 - 2u + a \right)^3 \left( au^2 + 2u + a \right)^3.

Now we see that the equation

\[ \text{Resultant}(\beta_2^+(u), p_2(u)) = \alpha \left( 1 - a^2 \right)^2 a^{48} \left( 1 + 2a^2 \right)^4 \left( 8a^4 - 4a^2 - 3 \right)^4 = 0, \]

with \( \alpha \neq 0 \), has a unique solution \( a_u = \sqrt{\frac{1 + \sqrt{10}}{2}} \) and its corresponding equilibrium is given by replacing this value into (35), in this way we obtain the position of the equilibrium

\[ u_3 = \frac{1}{3} \sqrt{11 + 4\sqrt{7}} + 2 \sqrt{2 \left( 19 + 11\sqrt{7} \right)}. \]

Replacing these two values into \( \beta_2 \), we obtain \( \beta_2 \approx 15.3865... \), thus \( \beta_2 > 0 \) along \( L_{u_3} \). To study the sign of \( \beta_2 \) along \( L_{u_1} \) and \( L_{u_2} \), we consider the second case \( u \in (a_1^2, 1) \) (both equilibria belong the upper hemisphere). Thus, reducing (47) to its polynomial form \( \beta_2^+(u) \) and using the fact that the equilibrium points \( L_{u_1} \) and \( L_{u_2} \) are contained in the set of roots of the polynomial \( p_4(u) \) defined in (36), we proceed as in the previous case to obtain

\[ 0 = \text{Resultant}(\beta_2^-(u), p_4(u)) = \alpha a^{240} (1 - a^2)^{120} f_1(a)^4 f_2(a)^4, \]

where \( \alpha \) is a nonzero constant and

\[ f_1(a) = 2048a^{16} - 2048a^{14} - 1280a^{12} + 1408a^{10} + 480a^8 - 192a^6 - 8a^4 + 60a^2 + 9, \]
\[ f_2(a) = 524288a^{24} - 3145728a^{22} + 8257536a^{20} - 12451840a^{18} + 11907072a^{16} - 7520256a^{14} + 3168256a^{12} - 871680a^{10} + 126816a^8 + 21952a^6 - 20928a^4 + 5592a^2 - 1073. \]

By using the Sturm criterion, we obtain that \( f_1(a) \) does not have real roots, while \( f_2(a) \) has exactly one real root in the interval \((0, 1)\), which is approximately \( a \approx 0.980199... \), however, the equilibria \( L_{u_1} \) and \( L_{u_2} \) exist only for \( a \in (0, a_1) \), thus, the corresponding curves of equilibria do not cross the curve \( \beta_2 = 0 \), and even more, \( \beta_2 > 0 \) along both equilibria.

Finally, we shall determine the sign of \( \beta_0 \). By virtue of Lemma 5.1, it is sufficient to study the sign of \( b_1 \). From condition \( f_{crit}(u; a) = 0 \), we obtain

\[ \frac{4 \omega^2 (1 - 8u^2 + 3u^4)}{\Delta^4} = \frac{(1 - 8u^2 + 3u^4)}{u(1 - u^2)} \left( \frac{\rho_1 l_1}{l_1^{5/2}} + \frac{\rho_2 l_2}{l_2^{5/2}} \right). \]
Replacing condition (49) into $b_1$, we get

$$b_1 = \frac{1}{u(1-u^2)} \left( \frac{(1-8a^2+3a^4)\rho_1 l_1+u(1-u^2)h_1}{l_1^{5/2}} + \frac{(1-8a^2+3a^4)\rho_2 l_2+u(1-u^2)h_2}{l_2^{5/2}} \right)$$

where $t_1 = (-t_{11}+t_{12})/l_1^{5/2}$, $t_2 = (t_{11}+t_{12})/l_2^{5/2}$, with

$$t_{11} = a \left( 1-u^2 \right) \left( a^2 u^8 + (28-44a^2) u^6 + (166a^2-136) u^4 + (28-44a^2) u^2 + a^2 \right),$$

$$t_{12} = 2\sqrt{1-a^2u} \left( 5a^2 u^8 + (12-60a^2) u^6 + (126a^2-40) u^4 + (12-60a^2) u^2 + 5a^2 \right).$$

Thus, the equation $b_1 = 0$ can be solved through the equation $t_1^2 - t_2^2 = 0$, which yields up the polynomial equation $\tilde{b}_1(u) = 0$, where

$$\tilde{b}_1(u) = a^4 u^{12} + (6a^4 - 24a^2) u^{10} + (15a^4 + 96a^2 - 48) u^8 + (20a^4 - 272a^2 + 160) u^6 + (15a^4 + 96a^2 - 48) u^4 + (6a^4 - 24a^2) u^2 + a^4.$$

Now, we compute Resultant($\tilde{b}_1(u)$, $p_4(u)$) for $a \in (0, a_1]$, which is proportional to the polynomial

$$a^{80} (1-a^2)^{40} (16777216a^{24} - 100663296a^{22} + 264241152a^{20} - 398458880a^{18} + 380436480a^{16} - 238288896a^{14} + 98435072a^{12} - 27525120a^{10} + 6138624a^8 - 1136128a^6 + 54336a^4 - 10560a^2 + 1511)^4. \quad (50)$$

Again, applying Sturm criterion, we obtain that (50) has a unique zero in $(0, a_1]$ and it is given by $a = a_1$. It is verified that $b_1(a_1, u(a_1)) = 0$ and defining $\delta_j = (a, u_j(a))$, where $L_{u_j} = (u_j(a), 0)$, for $j = 1, 2, 3$, we have that $\delta_1$ is contained inside the region $b_1(a, u) > 0$, for all $a \in (0, a_1)$, which implies $\beta_0 < 0$, while $\delta_2$ is contained inside the region $b_1(a, u) < 0$, for all $a \in (0, a_1)$, which implies $\beta_0 > 0$. For the curve $\delta_3$, we must solve the equation Resultant($\tilde{b}_1(u)$, $p_2(u)$) = 0 for $a \in (a_5, 1)$, where this resultant is proportional to the polynomial

$$a^{16} (1-a^2)^8 \left( 64a^6 - 48a^4 - 4a^2 + 1 \right)^4,$$

which has just one zero in the interval $(a_5, 1)$ given approximately by

$$a_s \approx 0.8964262021183806...$$

By evaluating $b_1$ at $(a_s, u_3(a_s))$, where $u_3$ is given in (35), we obtain $b_1(a_s, u_3(a_s)) \approx -0.0146411...$. Therefore, $a_s$ does not provide simultaneous solutions of (31) and $b_1 = 0$, i.e., the curve $\delta_3$ does not cross the boundary of the region $b_1 < 0$. In fact, $\delta_3$ is contained inside the region $b_1 < 0$, so, by Lemma 5.1, $\beta_0 < 0$ along $L_{u_3}$.

The last step consists in the determination of the sign of the discriminant $\delta = \beta_2^2 - 4\beta_0$ along the equilibrium $L_{u_2}$. For this equilibrium we already know that $b_1, b_2 < 0$ and $\beta_0, \beta_2 > 0$ for all $a \in (0, a_1)$. Since $b_1 + b_2 < 0$, then from definition of $\beta_2$ given in (41), we have $\beta_2 > -\Delta_3^2 (b_1 + b_2) > 0$, then
Table 3
Type of eigenvalues and stability for the collinear relative equilibria of the R3BP on $S^2$.

<table>
<thead>
<tr>
<th>Type</th>
<th>$\beta_2$</th>
<th>$\beta_0$</th>
<th>$\delta$</th>
<th>$x$</th>
<th>$\lambda$</th>
<th>Type of stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{u1}$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$x_1 &lt; 0 &lt; x_2$</td>
<td>$\pm i \sqrt{</td>
<td>x_1</td>
</tr>
<tr>
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<td>$&gt; 0$</td>
<td>$0$</td>
<td>$x_1 &lt; x_2 &lt; 0$</td>
<td>$0, 0, \pm i \sqrt{\beta_0^2}$</td>
<td>Spectrally stable</td>
</tr>
<tr>
<td>$L_{u12}$</td>
<td>$0$</td>
<td>$= 0$</td>
<td>$0$</td>
<td>$-\beta_2, 0$</td>
<td>$0, 0, \pm i \sqrt{\beta_0^2}$</td>
<td>Linearly stable</td>
</tr>
<tr>
<td>$L_{u3}$</td>
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<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$x_1 &lt; 0 &lt; x_2$</td>
<td>$\pm i \sqrt{</td>
<td>x_1</td>
</tr>
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</table>

Table 4
Stability of the north pole.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$(0, \sqrt{\frac{3}{2}})$</th>
<th>$\sqrt{\frac{3}{2}}$</th>
<th>$(\sqrt{\frac{3}{2}}, \sqrt{\frac{7}{5}})$</th>
<th>$\sqrt{\frac{7}{5}}$</th>
<th>$(\sqrt{\frac{7}{5}}, 1)$</th>
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<td>Unstable</td>
<td>Linearly unstable</td>
<td>Linearly stable</td>
<td>Linearly unstable</td>
<td>Unstable</td>
</tr>
</tbody>
</table>

Table 5
Stability of the south pole.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$(0, \sqrt{\frac{3}{2}})$</th>
<th>$\sqrt{\frac{3}{2}}$</th>
<th>$(\sqrt{\frac{3}{2}}, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Unstable</td>
<td>Linearly unstable</td>
<td>Linearly stable</td>
</tr>
</tbody>
</table>

$\delta > \frac{\Lambda^4}{16} \left( (b_1 + b_2)^2 - 4 \beta_0 \right) = \frac{\Lambda^4}{16} (b_1^2 + b_2^2) - \frac{\Lambda^4}{8} b_1 b_2 = \frac{\Lambda^4}{16} (b_1 - b_2)^2 \geq 0$.

In order to summarize our analysis, putting $x = \lambda^2$, the characteristic polynomial takes the form

$\tilde{p}(x) = x^2 + \beta_2 x + \beta_0$.

Let $x_1$ and $x_2$ be their roots. We describe the above analysis in Table 3. We observe that when $a = a_1$, then $\beta_0 = 0$, therefore the equilibrium $L_{u12}$ has a double zero eigenvalue and a pair of opposite pure imaginary eigenvalues, which implies spectral stability. This finishes the proof of Theorem 5.3. ∎

For the case of negative curvature, we have the following result whose proof is easier.

**Theorem 5.4.** For $\sigma = -1$, the collinear equilibria are unstable, even more, they are center-saddle.

**Proof.** For $\sigma = -1$, it is easy to verify that $h_1, h_2 > 0$, then $b_1 > 0$ and by Lemma 5.1, we have that $\beta_0 < 0$, and the result follows. ∎

We finish this section by studying the linear stability of the poles.

**Proposition 5.1.** For $\sigma = 1$, the linear stability of the north and south poles depends on the parameter $a$ and it is given by Tables 4 and 5, respectively. For $\sigma = -1$, the pole is unstable for all $a > 0$. 


Proof. The linearization matrix at \( (0, 0, 0, 0) \) is given by

\[
\mathcal{A}_\epsilon^\sigma = \begin{pmatrix}
0 & \omega & -\omega & 0 \\
-\omega & 0 & 0 & 0 \\
\frac{\sqrt{1-a^2\sigma}}{\omega} & 0 & 0 & -\omega \\
0 & \frac{\sqrt{1-a^2\sigma}}{\omega} & 0 & -\omega
\end{pmatrix}
\]

with eigenvalues \( \pm \lambda, \pm \rho \), where

\[
\lambda = \omega \sqrt{f^\sigma_\epsilon (a)}, \quad \rho = \omega \sqrt{g^\sigma_\epsilon (a)},
\]

\[
f^\sigma_\epsilon = a^\sigma_\epsilon + 4(1-\sigma a^2)\sqrt{\beta^\sigma_\epsilon}, \quad g^\sigma_\epsilon = a^\sigma_\epsilon - 4(1-\sigma a^2)\sqrt{\beta^\sigma_\epsilon},
\]

\[
a^\sigma_\epsilon = -1 - 4\epsilon (1-\sigma a^2)^2, \quad \beta^\sigma_\epsilon = \epsilon + 9(1-\sigma a^2)^2.
\]

For \( S^2 \), the north pole corresponds to the origin with \( \epsilon = -1 \) and \( \sigma = 1 \). It is verified that \( \beta^+ = (2 - 3a^2)(4 - 3a^2) > 0 \), if and only if, \( a \in (0, \sqrt{2/3}) \) and in this interval \( g^+(a) < 0 \), while \( f^+(a) > 0 \), if and only if, \( a \in I_1 = (0, \sqrt{4 - \sqrt{2}/2}) \), then defining the intervals \( I_2 = (\sqrt{4 - \sqrt{2}/2}, \sqrt{2/3}) \) and \( I_3 = (\sqrt{2/3}, 1) \), we have that the eigenvalues associated to the north pole of \( S^2 \) are characterized in Table 6. For the cases \( a = \frac{\sqrt{4 - \sqrt{2}}}{2} \) and \( a = \sqrt{2/3} \), it is verified that the linearization matrices are not diagonalizable, thus, in both cases, the north pole is linearly unstable.

For \( S^2 \), the south pole corresponds to the origin with \( \epsilon = 1 \) and \( \sigma = 1 \). It is verified that \( \beta^- = 9(1 - a^2)^2 + 1 > 0 \) and \( g^-(a) < 0 \) for all \( a \in (0, 1) \), while \( f^+(a) > 0 \), if and only if, \( a \in I_1 = (0, \sqrt{3}/2) \). Defining the interval \( J_1 = (\sqrt{3}/2, 1) \), we have that the eigenvalues associated to the south pole of \( S^2 \) are characterized in Table 7. For the case \( a = \frac{\sqrt{3}}{2} \), it is verified that the linearization matrix is not diagonalizable, therefore, the south pole is linearly unstable.
For \( \mathbb{H}^2 \), studying the pole corresponds to the origin with \( \epsilon = -1 \) and \( \sigma = -1 \). It is verified that
\[
f_\pm(a) = \left(2a^2 + 1\right) \left(2a^2 + 3\right) + 4 \left(a^2 + 1\right) \sqrt{(3a^2 + 2)(3a^2 + 4)} > 0,
\]
for all \( a > 0 \) and
\[
f_\pm(a) g_\pm(a) = -\left(8a^4 + 16a^2 + 7\right) \left(16a^4 + 32a^2 + 17\right) < 0,
\]
thus, \( g_\pm(a) < 0 \) for all \( a > 0 \), then \( \lambda \in \mathbb{R}^+ \) and \( \rho \in i\mathbb{R}^+ \) for all \( a > 0 \), which completes the proof. \( \square \)

6. Periodic solutions for the symmetric R3BP

In this section, we shall prove the existence of periodic orbits for the symmetric R3BP on surfaces of constant curvature in two different ways. We start by looking periodic orbits near the poles of the respective surface, as a way of exploiting the nature of the eigenvalues associated to the poles, which as we have already proved, are equilibrium points of the symmetric problem for any value of the parameter \( a \). On the other hand, we will make use of averaging theory for perturbed Hamiltonian systems to obtain periodic orbits close to an integrable Hamiltonian system.

6.1. Periodic orbits near the poles

One of the most famous theorems concerning the existence of periodic solutions of ordinary differential equations is the Lyapunov Center Theorem [25]. In the Hamiltonian case assumes that the origin is an equilibrium point of elliptic type. The Lyapunov Center Theorem states that if one eigenvalue is purely imaginary \( \lambda_1 = i\alpha \), and the quotients \( \lambda_j/\lambda_1 \) for \( j = 2, 3, \cdots, l \) are not integer numbers, then there exist a family of periodic orbits emanating from the equilibrium whose minimal period tends to \( 2\pi/|\alpha| \). The proof of this theorem can be found in [23]. In this section we will exploit the fact that for certain intervals of the parameter \( a \), the eigenvalues corresponding to the poles are purely imaginary, this fact give us chance to find periodic orbits by applying the Lyapunov Center Theorem. Let
\[
a_m^\pm = \frac{1}{2} \left[ 4 - \sqrt{\frac{\pm(m^4 + 6m^2 + 1) + (m^2 + 1) \sqrt{9m^4 - 2m^2 + 9}}{2m^4 + 5m^2 + 2}} \right], \ m \in \mathbb{Z}
\]
be a sequence of values of the radial parameter \( a \).

**Theorem 6.1.** For the parameter \( a \), assume that all eigenvalues corresponding to the poles are purely imaginaries, then for the case of positive curvature (on \( S^2 \)), the following statements hold:

i) If \( a \in I_1 \cup I_2 \), then there is a family of \( \tau(\nu) \)-periodic orbits \( x(t, \nu) \), emanating from the north pole, such that \( \lim_{\nu \to 0} \tau(\nu) = \frac{2\pi}{\omega \sqrt{-g^\pm(a)}} \). Even more, for \( a \in I_2 \) and \( a \neq a_m^+ \), for all integer \( m > 1 \), there is another family of \( \tau(\nu) \)-periodic orbits, with \( \lim_{\nu \to 0} \tau(\nu) = \frac{2\pi}{\omega \sqrt{-f^\pm(a)}} \).
\( \text{ii) If } a \in J_1, \text{ then there is a family of } \tau(v)\text{-periodic orbits } x(t, v) \text{, emanating from the south pole, such that, } \lim_{v \to 0} \tau(v) = \frac{2\pi}{\omega \sqrt{-g^{-}(a)}}. \)

\( \text{iii) If } a \in J_2, \text{ then there exists a family of } \tau(v)\text{-periodic orbits } x(t, v) \text{, emanating from the south pole, such that, } \lim_{v \to 0} \tau(v) = \frac{2\pi}{\omega \sqrt{-g^{+}(a)}}. \text{ Even more, for } a \in J_2 \text{ and } a \neq a_m, \text{ for all integer } m > 1, \text{ there is another family of } \tau(v)\text{-periodic orbits, with } \lim_{v \to 0} \tau(v) = \frac{2\pi}{\omega \sqrt{-f^{+}(a)}}. \)

For the case of negative curvature, that is on \( \mathbb{H}^{2} \), there is a family of \( \tau(v)\)-periodic orbits \( x(t, v) \), emanating from the pole, such that \( \lim_{v \to 0} \tau(v) = \frac{2\pi}{\omega \sqrt{-g^{-}(a)}}. \)

**Proof.** To prove part i), we observe in Table 6, that the intervals where the Lyapunov Center Theorem might be applied are \( I_1 = (0, \sqrt{4 - \sqrt{2}}/2) \) and \( I_2 = (\sqrt{4 - \sqrt{2}}/2, \sqrt{2}/3) \). So, when \( a \in I_1 \), the pure imaginary eigenvalue is \( \rho = i\omega \sqrt{-g^{-}(a)} \) and due to the fact that \( \lambda \in \mathbb{R}^{+} \), the condition \( \lambda/\rho \neq \mathbb{Z} \) holds for all \( a \in I_1 \).

For \( a \in I_2 \), we know that \( \lambda, \rho \in i\mathbb{R}^{+} \), with \( g^{+}(a) < f^{+}(a) < 0 \), then

\[
0 < \left( \frac{\lambda}{\rho} \right)^{2} = \frac{|f^{+}(a)|}{|g^{+}(a)|} < 1, \tag{53}
\]

and it follows that \( \lambda/\rho \notin \mathbb{Z} \) for all \( a \in I_2 \).

Now, in order to find a new family of periodic orbits we must analyze the reciprocal of the previous ratio. We obtain from (53) that \( \rho/\lambda > 1 \), which means that resonances might occur for certain values of \( a \), i.e., it is necessary to determine which values of the parameter \( a \) satisfy the condition \( \rho/\lambda \notin \mathbb{Z} \). Let \( m \in \mathbb{Z} \) with \( m > 1 \), by solving the equation (\( \rho/\lambda \))\(^{2} = m^{2} \), we obtain the solution \( a = a_{m}+ \).

By straightforward computations it is easy verify that \( a_{m} \in I_2 \) for all integer \( m > 1 \). The ends of the interval \( I_2 \) are reached when \( m = 1 \) and when \( m \to +\infty \), that is \( a_{1}+ = \sqrt{2}/3 \) and \( \lim_{m \to \infty} a_{m}+ = \sqrt{4 - \sqrt{2}}/2 \), here we have resonances. Thus, for \( a \in I_2 \), \( \rho/\lambda \notin \mathbb{Z} \), and if and only if, \( a \neq a_{m}+ \), for all \( m \in \mathbb{Z}, m > 1 \), which assures the existence of the other family of periodic orbits and the proof of part i) is completed.

To prove part ii), we see from Table 7, that Lyapunov Center Theorem might be applied for any \( a \in (0, 1) \) such that \( a \neq \sqrt{3}/2 \). For any \( a \in J_1 \), the condition \( \lambda/\rho \notin \mathbb{Z} \) is immediately verified and part ii) follows.

For the interval \( J_2 \), both eigenvalues \( \lambda \) and \( \rho \) are pure imaginary and also it is satisfied \( g^{+}(a) < f^{+}(a) < 0 \), which implies immediately that \( \lambda/\rho \notin \mathbb{Z} \), which gives the existence of the family of periodic orbits. Since \( \rho/\lambda > 1 \), it is necessary to discard the values of the parameter \( a \) such that \( \rho/\lambda \in \mathbb{Z} \). Let \( m > 1 \) an integer, then the equation (\( \rho/\lambda \))\(^{2} = m^{2} \) gives the solution \( a = a_{m}^{-} \) and it is verified that \( a_{m}^{-} \in J_2 \) for all integer \( m > 1 \), \( \lim_{m \to 1} a_{m}^{-} = 1 \) and \( \lim_{m \to \infty} a_{m}^{-} = \sqrt{3}/2 \). Thus, for \( a \in J_2 \), \( a \notin \mathbb{Z} \), if and only if, \( a \neq a_{m}^{-} \), for all \( m \in \mathbb{Z}, m > 1 \), where it follows the existence of the other family of periodic orbits, which proves part iii).

The proof for the case of negative curvature follows similar to part i), by using the fact that \( \lambda \in \mathbb{R}^{+} \) and \( \rho \in i\mathbb{R}^{+} \), which completes the proof of Theorem 6.1. \( \square \)
6.2. Periodic orbits close to an integrable Hamiltonian

We can also obtain periodic orbits by using perturbation theory, i.e., we can continue periodic orbits from an integrable Hamiltonian system through the appropriate introduction of a perturbing parameter and the application of Reeb’s Theorem [27].

Let us introduce the perturbation parameter \( \varepsilon \) in the symmetric problem by means of scaling the parameter \( a \) as \( a \mapsto \varepsilon^2 a \) and considering the time-scaling \( dt = \varepsilon^3 d\tau \). So, for small values of the parameter \( \varepsilon \), the primaries rotate near the north pole (in the case of \( \mathbb{S}^2 \)) or near the vertex (in the case of \( \mathbb{H}^2 \)). In other words, this means that we are considering a perturbation of the curved Kepler problem (see [1], [4]). So, after applying the previous two transformations, we expand into power series over \( \varepsilon = 0 \) and introducing polar coordinates \( (\rho, \theta, R, \Theta) \) given by the symplectic transformation

\[
\begin{align*}
 u &= \rho \cos \theta, & v &= \rho \sin \theta, & p_u &= R \cos \theta - \frac{\Theta}{\rho} \sin \theta, & p_v &= R \sin \theta + \frac{\Theta}{\rho} \cos \theta,
\end{align*}
\]

we obtain

\[
\mathcal{H}_\varepsilon = \mathcal{H}_0(\Theta) + \varepsilon^2 \mathcal{H}_3(\rho, R, \Theta) + \mathcal{O}(\varepsilon^4),
\]

where

\[
\mathcal{H}_0 = \alpha_1 \Theta, \quad \mathcal{H}_3 = \frac{\Theta^2 (1 + \sigma \rho^2)^2 + \rho \left( R^2 \rho (1 + \sigma \rho^2)^2 + 4\sigma (\rho^2 - \sigma) \right)}{8\rho^2},
\]

and \( \alpha_1 = -\frac{1}{2\sqrt{2a^{3/2}}} \). We call the Hamiltonian (54) as the limit problem.

**Theorem 6.2.** Let \( h \) be a fixed constant and fix \( \mathcal{H}_0 = h \). For the Hamiltonian system associated to the Hamiltonian function (54) and for any admissible value of the parameter \( a \) and for \( \varepsilon \) small enough, there is a one-parameter family of \( \bar{T}(\varepsilon) \approx \frac{4\pi \sqrt{2a^{3/2} \varepsilon}}{h} + \mathcal{O}(\varepsilon^6) \)-periodic solutions which are strongly stable.

**Proof.** Fixing \( h = \mathcal{H}_0 \), the flow associated to the unperturbed Hamiltonian \( \mathcal{H}_0 \) is given by

\[
\phi^t_0(\rho, \theta, R, \Theta) = (\rho, \alpha_1 t + \Theta, R, \Theta), \quad \text{with} \quad \Theta = \frac{h}{\alpha_1}.
\]

In order to apply Reeb’s theorem (see [13], [24], [27]), first we compute the average of \( \mathcal{H}_3 \) over the flow \( \phi^t_0 \), to obtain

\[
\mathcal{H}_3 = \frac{1}{T} \int_0^T \mathcal{H}_3(\phi^s_0) \, ds = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_3 \, d\theta = \mathcal{H}_3.
\]

Note that, averaging the first perturbation is equivalent to normalize up to the first perturbation. Now we compute the critic points of \( \mathcal{H}_3 \) by solving \( \nabla_{(\rho, R)} \mathcal{H}_3 = (0, 0) \), and we get that there is a unique critical point, given by
\[ \rho_0 = \frac{\sigma(\sqrt{1 + \sigma \Theta^4} - 1)}{\Theta^2}, \quad R_0 = 0, \] 

(55)

with \( \Theta \neq 0 \) for \( \sigma = 1 \) and \( 0 < |\Theta| < 1 \) for \( \sigma = -1 \). We verify that the matrix

\[
A = D^2 \mathcal{H}_3(\rho_0, R_0) = \begin{pmatrix}
0 & \frac{(1 + \sigma \Theta^4 - \sqrt{1 + \sigma \Theta^4})^2}{\Theta^8} \\
\Theta^2(1 + \sigma \Theta^4) & 0
\end{pmatrix}
\]

has determinant \( \frac{(1 + \sigma \Theta^4)^2}{\Theta^8} \neq 0 \), then, by applying Reeb’s Theorem (see [27]), it follows that for each \( \varepsilon \) small enough, there is a \( T(\varepsilon) \)-periodic solution of (54)

\[
\varphi(t, \varepsilon) = (\rho(t, \varepsilon), \theta(t, \varepsilon), R(t, \varepsilon), \Theta(t, \varepsilon)),
\]

such that

\[
\lim_{\theta \to 0} \varphi(t, \varepsilon) = (\rho_0, \theta_0, 0, h/\alpha_1).
\]

Since, the eigenvalues of \( A \) are given by \( \lambda_{\pm} = \pm \delta i \), with

\[
\delta = \frac{(1 + \sigma \Theta^4) \sqrt{\delta_1 - \delta_2}}{\Theta^3 \left( \sqrt{1 + \sigma \Theta^4} - 1 \right)^2}, \quad \delta_1 = \Theta^8 + 8\sigma \Theta^4 + 8, \quad \delta_2 = 4\sqrt{1 + \sigma \Theta^4} \left( 2 + \sigma \Theta^4 \right),
\]

it is easy to verify that \( \delta > 0 \) for all \( \theta \neq 0 \) in the case \( \sigma = 1 \) and \( 0 < |\Theta| < 1 \) in the case \( \sigma = -1 \), which shows that, in fact, \( \lambda_{\pm} \in i\mathbb{R} \) and then, in virtue of [27], we obtain that the characteristic multipliers are given by

\[
1, 1, 1 \pm \frac{2\pi \alpha_1 i \delta}{h} + O(\varepsilon^4),
\]

which implies that the periodic orbit is strongly stable (see [29]). Finally, we approximate the period \( \bar{T}(\varepsilon) \) as

\[
\bar{T}(\varepsilon) = \frac{2\pi}{\alpha_1 + \varepsilon^3 \frac{\Theta(1 + \sigma \rho^2)^2}{4\rho^4} + O(\varepsilon^4)} = \frac{2\pi}{\alpha_1} - \varepsilon^3 \frac{2\pi \left( \alpha_1^4 + \sigma h^4 \right)}{\alpha_1^3 h^3} + O(\varepsilon^4),
\]

and considering the initial time-scaling \( t = \varepsilon^3 \tau \), we obtain that the period is approximated by \( T(\varepsilon) = \varepsilon^{-3} \bar{T}(\varepsilon) \), which completes the proof. \( \square \)

7. Existence of KAM tori for the symmetric R3BP

Using the results related to periodic orbits of the previous section, we shall prove the existence of invariant KAM tori by using a classic Arnold’s result for isoenergetically non-degeneracy condition (see [2]) as well as a more current result proved by Han, Li and Yi in [15].
Table 8  
Values of the parameter $a$ such that resonances up to order 4 appear.  

<table>
<thead>
<tr>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(k_1, k_2)$</td>
<td>$(1, 0)$</td>
<td>$(1, -1)$</td>
<td>$(2, -1)$</td>
</tr>
<tr>
<td>$a_N^{(j)}$</td>
<td>$\frac{\sqrt{4 - \sqrt{2}}}{2}$</td>
<td>$\sqrt{\frac{2}{3}}$</td>
<td>$\frac{1}{6} \sqrt{36 - \frac{3}{2} (41 + 5 \sqrt{145})}$</td>
</tr>
<tr>
<td>$a_S^{(j)}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{6} \sqrt{36 - \frac{3}{2} (5 \sqrt{145} - 41)}$</td>
<td>$1 - \frac{1}{\sqrt{17}} \left(15 \sqrt{3} - 17\right)$</td>
</tr>
</tbody>
</table>

7.1. Invariant KAM tori near the poles in the symmetric R3BP on $S^2$

We consider the symmetric R3BP on $S^2$ for values of the parameter $a$ where the eigenvalues of the poles are pure imaginary and there are no resonances up to order 4. If $k = (k_1, k_2) \in \mathbb{Z}^2$ is the vector of resonance of order $j$, then, from (52), we must exclude the values of $a$ such that $k_1^2 f_1^j (a) = k_2^2 g_1^j (a)$, with $|k_1| + |k_2| = j$, $j = 1, 2, 3, 4$. If we denote by $a_N^{(j)}$ and $a_S^{(j)}$ such values of $a$ generating a resonance of order $j$ for the north and south poles, respectively, then they are given by the Table 8.

Now, we unified the set where we are going to consider the value of the parameter $a$ for both poles.

\[
J = \begin{cases} 
\left(\frac{\sqrt{4 - \sqrt{2}}}{2}, \sqrt{\frac{2}{3}}\right) \backslash \{a_N^{(4)}\} & \text{if } \epsilon = -1, \\
\left(\frac{\sqrt{3}}{2}, 1\right) \backslash \{a_S^{(4)}\} & \text{if } \epsilon = 1. 
\end{cases}
\]

Theorem 7.1. For every $a \in J$, the Hamiltonian system associated to the symmetric problem in a neighbourhood of each pole has invariant tori close to the linearized system, these tori form a set whose relative measure in the polydisk $|f| < \epsilon$ tends to 1 as $\epsilon \to 0$ (in fact, this measure is at least $1 - O(\epsilon^{1/4})$). In both cases such tori occupy a larger part of each energy level passing near the poles.

Proof. We are going to prove that our Hamiltonian is close to an integrable system which is isoenergetically non-degenerate given in [2]. For this purpose, we need to normalize our Hamiltonian up to order 4, such that the truncated Hamiltonian up to order 4 is integrable, i.e., it only depends on the actions. This could be true if we exclude the values of $a$ where a resonance of order $\leq 4$ appears. Since the model of the problem the Hamiltonian (19) has the symmetries

\[(u, v, p_u, p_v) \to (-u, v, p_u, -p_v) \text{ and } (u, v, p_u, p_v) \to (u, -v, -p_u, p_v),\]

the terms of odd order in the Taylor expansion around the origin do not appear. Thus, resonances of order 3 do not affect the terms of the normal form up to 4. So, it will be enough to take $a \in J$.

If $(X, Y, P_1, P_2)$ are the coordinates that normalize the quadratic part of the Taylor expansion, then by using the Markeev’s algorithm (see [21]), we obtain that the Hamiltonian function can be written as

\[\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_2 + \ldots,\]  

(56)
where
\[ H_0 = -\frac{\omega_1}{2} (X^2 + P_1^2) + \frac{\omega_2}{2} (Y^2 + P_2^2), \]
with
\[ \omega_1 = \omega \sqrt{-f'(a)}, \quad \omega_2 = \omega \sqrt{-g'(a)}, \]
and the quartic part \( H_2 \) takes the form
\[ H_2 = \sum_{k_1+k_2+k_3+k_4=4} d_{k_1k_2k_3k_4} X^{k_1} Y^{k_2} P_1^{k_3} P_2^{k_4}, \tag{57} \]
with
\[ d_{2110} = d_{2101} = d_{1210} = d_{1201} = d_{0121} = d_{0112} = d_{0112} = d_{0121} = 0, \]
\[ d_{3010} = d_{3001} = d_{0310} = d_{0103} = d_{0301} = d_{0130} = 0, \tag{58} \]
and the remaining coefficients are given in Appendix A for the north pole and in Appendix B for the south pole. Now, we follow the Lie method described in [5] and [23] to obtain that the normalized quartic part is given by the homogeneous fourth degree polynomial
\[ H_2 = \sum_{k_1+k_2+k_3+k_4=4} D_{k_1k_2k_3k_4} X^{k_1} Y^{k_2} P_1^{k_3} P_2^{k_4}, \tag{59} \]
where, in virtue of the Lie–Deprit equation
\[ \{H_0, H_j\} = 0, \quad j = 1, 2, \ldots \tag{60} \]
we obtain that the coefficients of the normalized quartic part satisfy
\[ D_{0013} = D_{0031} = D_{0103} = D_{0112} = D_{0121} = D_{0211} = D_{0301} = D_{0310} = 0, \]
\[ D_{1003} = D_{1012} = D_{1021} = D_{1030} = D_{1102} = D_{1111} = D_{1120} = D_{1201} = D_{1210} = 0, \]
\[ D_{1300} = D_{2011} = D_{2101} = D_{2110} = D_{3001} = D_{3010} = D_{3100} = 0. \tag{61} \]
Thus, \( H_2 \) has the form
\[ H_2 = D_{4000} X^4 + D_{0400} Y^4 + D_{0040} P_1^4 + D_{0004} P_2^4 + D_{2200} X^2 Y^2 + \]
\[ D_{2020} X^2 P_1^2 + D_{2002} X^2 P_2^2 + D_{0220} Y^2 P_1^2 + D_{0202} Y^2 P_2^2 + \]
\[ D_{0022} P_1^2 P_2^2, \tag{62} \]
while, in action-angle variables, \( H_2 \) is given by
\[ H_2 = \omega_1 I_1^2 + \omega_1 I_1 I_2 + \omega_2 I_2^2, \tag{63} \]
with
Fig. 3. Isoenergetically non-degeneracy condition.

\[
\omega_{11} = 3D_{0040} + D_{2020} + 3D_{4000}, \\
\omega_{12} = 2D_{0022} + D_{0220} + D_{2002} + D_{2200}, \\
\omega_{22} = 3D_{0004} + D_{0202} + 3D_{0400}.
\]

By the Lie–Deprit normal form algorithm we obtain the remaining terms of the normalized quartic part

\begin{align*}
D_{4000} &= \frac{1}{4}(3d_{0040} + d_{2020} + 3d_{4000}), \\
D_{0040} &= \frac{1}{4}(3d_{0004} + d_{0202} + 3d_{0400}), \\
D_{2200} &= \frac{1}{4}(d_{0022} + d_{0220} + d_{2002} + d_{2200}), \\
D_{2002} &= \frac{1}{2}(d_{0022} + d_{0220} + d_{2002} + d_{2200}), \\
D_{0202} &= \frac{1}{2}(3d_{0004} + d_{0202} + 3d_{0400}).
\end{align*}

These calculations were made using Mathematica 9®.

Now, the Hamiltonian of the symmetric problem can be written as

\[
\mathcal{H} = \mathcal{H}^0 + \cdots, \quad \mathcal{H}^0 = -\omega_1 I_1 + \omega_2 I_2 + \omega_{11} I_1^2 + \omega_{12} I_1 I_2 + \omega_{22} I_2^2.
\]

Here the dots denote terms of order higher than four with respect to the distance from the origin. The isoenergetically non-degeneracy condition is given by

\[
\Omega(a) = \det \begin{pmatrix}
\frac{\partial^2 \mathcal{H}^0}{\partial I_1^2} & \frac{\partial \mathcal{H}^0}{\partial I_1} & \frac{\partial \mathcal{H}^0}{\partial I_2} \\
\frac{\partial \mathcal{H}^0}{\partial I_1} & \omega_{11} & \omega_{12} \\
\frac{\partial \mathcal{H}^0}{\partial I_2} & \omega_{12} & \omega_{22} \\
0 & \omega_1 & \omega_2
\end{pmatrix} = -2\mathcal{H}^2(\omega_2, \omega_1) \neq 0.
\]

We observe in Fig. 3 that the condition (65) is in fact verified and the result follows from Theorem 6.23 in [2]. □
7.2. Invariant KAM close to an integrable Hamiltonian

Let us add the term of order 4 of the Taylor expansion in (54) to obtain the Hamiltonian

\[ H_\varepsilon = H_0 + \varepsilon^3 H_3 + \varepsilon^4 H_4 + O(\varepsilon^7), \]

where \( H_4 = \alpha_2 \Theta, \alpha_2 = -\sigma \frac{3\sqrt{2}}{8\sqrt{2}}. \)

**Theorem 7.2.** Around the periodic solutions (given by Theorem 6.2) there are families of invariant KAM 2-tori for the symmetric R3BP on \( S^2 \) and \( H^2 \), of the full system associated to the Hamiltonian (19). The complement of the set in which the KAM tori persists has measure of order \( O(\varepsilon^8) \). Even more, the near circular periodic solutions are orbitally stable and enclosed by invariant KAM 2-tori for \( \varepsilon \) small enough.

**Proof.** Let us consider the \( \varepsilon^{-2} \)-symplectic transformation

\[ Q = \varepsilon^{-1}(\rho - \rho_0), \quad P = \varepsilon^{-1} R, \]

where \( \rho_0 \) is the value given in (55), together with the time-scaling \( t \mapsto \varepsilon t \) and expanding in power series of \( \varepsilon \) we obtain

\[ H_3 = H_{30} + \varepsilon^2 H_{32} + O(\varepsilon^3), \]

where

\[ H_{30} = -\frac{1 + \sigma \Theta^4}{2\Theta^2}, \quad H_{32} = AQ^2 + BP^2, \]

with

\[ A = \frac{\Theta^2(1+\sigma \Theta^4)}{2(1-\sqrt{1+\sigma \Theta^4})^2}, \quad B = \frac{(1+\sigma \Theta^4)(1-\sqrt{1+\sigma \Theta^4})^2}{2\Theta^8}. \]

Now, we introduce the action-angle coordinates \((I_2, \psi)\) through

\[ Q = \sqrt[4]{\frac{B}{A}} \sqrt{2I_2} \cos \psi, \quad P = \sqrt[4]{\frac{A}{B}} \sqrt{2I_2} \sin \psi, \]

which satisfies \( dQ \wedge dP = dI_2 \wedge d\psi \) and defining \( I_1 \equiv \Theta \), we obtain that (66) becomes

\[ H = \alpha_1 I_1 - \varepsilon^3 \frac{1 + \sigma I_1^4}{2I_1^2} + \varepsilon^4 \alpha_2 I_1 + \varepsilon^5 \left( \frac{1}{I_1} + \sigma I_1 \right) I_2 + O(\varepsilon^6). \]

(67)

Now, we apply the main Theorem of [15] (see also Theorem 2.4 in [24]) with \( a = 3, m_1 = 3, m_2 = 4, m_3 = 5, n_0 = n_1 = n_2 = 1, n_3 = 2, I^{n_0} = I^{n_1} = I^{n_2} = (I_1), I^{n_3} = (I_1, I_2), \bar{I}^{n_0} = \bar{I}^{n_1} = \bar{I}^{n_2} = (I_1) \) and \( I^{n_3} = (I_2) \). Then we obtain the vector
\[ \Omega(I) = \left( \alpha_1, \sigma I_1 + \frac{1}{I_1^3}, \alpha_2, \sigma I_1 + \frac{1}{I_1^3} \right)^T, \]

and the matrix whose columns are \( \Omega(I), \partial I_1 \Omega(I) \) and \( \partial I_2 \Omega(I) \) (here we choose the integer \( s = 1 \) of the main Theorem of [15]) is given by

\[
M = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
\sigma I_1 + \frac{1}{I_1^3} & \sigma - \frac{3}{I_1^3} & 0 \\
\alpha_2 & 0 & 0 \\
\sigma I_1 + \frac{1}{I_1^3} & \sigma - \frac{3}{I_1^3} & 0
\end{pmatrix},
\]

which clearly has rank 2. Additionally, since \( s = 1 \) and \( b = \sum_{i=1}^{a} = m_i(n_i - n_{i-1}) = 8 \), according to [15], the excluding measure for the existence of quasi-periodic invariant tori is of order \( O(\varepsilon^b) \), which in our case is \( O(\varepsilon^8) \). This completes the proof of Theorem 7.2. \( \square \)

Note that due to the high degeneracy of the Hamiltonian (67), we cannot apply Theorem 6.23 of [2] as in the proof of Theorem 7.1, for this reason, we need to make use of Han, Li and Yi’s Theorem.

8. Conclusions

In this work we have studied a particular restricted three body problem on surfaces of constant curvature. Using the stereographic projection onto the equatorial plane we work on the curved plane and on the hyperbolic space. We consider an elliptic relative equilibrium of the curved two body problem which in rotating coordinates plays the role of the primaries. A complete study related to bifurcations on the number of equilibria was carried out for the particular case when the primaries move on the same parallel. In this case, the curved problem depends on one parameter given by the radius \( \alpha \) (the radius of the circle on the same parallel), while the analogous planar case does not depend on any parameter. For negative curvature the number of relative equilibria do not present bifurcations, its number is the same as in the planar case, i.e., there are three collinear and two triangular relative equilibria (that we have called them as isosceles equilibria by short). In this case we have proved that for \( \mu = 1/2 \), all of them are unstable, as well as in the planar case. The case of positive curvature differs significantly from the planar case. In fact, we found five bifurcation values with respect to the parameter \( a \in (0, 1) \); nevertheless in none of the intervals determined by such values of the parameter the number of equilibria coincides with the ones for the planar case. In addition, we proved the existence of periodic orbits surrounding the poles (or vertex) through the application of the Lyapunov Center Theorem for certain intervals of the parameter and also the existence of KAM tori related to these periodic orbits for the case of \( S^2 \), where we were able to write the Hamiltonian function as a perturbation of an integrable Hamiltonian system satisfying the isoenergetically non-degeneracy condition.

On the other hand, we have applied Averaging Theory to prove the existence of periodic orbits after to introduce an appropriate small perturbation parameter, which becomes the Hamiltonian system close to the curved Kepler problem. Finally, these periodic orbits allowed us to prove the existence of KAM tori surrounding them through the use of Han, Li and Yi’s Theorem.
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Appendix A. Coefficient for \( \mathcal{H}_2 \) at the north pole

Defining the quantities

\[
m_1 = \omega, \quad m_2 = \frac{8\sqrt{1 - a^2}}{a^3}, \tag{68}
\]

\[
\alpha_1 = 9m_2 - 32m_1^2,
\]

\[
\alpha_2 = -9m_2^4 + \sqrt{81m_2^8 + 768m_2^6},
\]

\[
\alpha_3 = \sqrt{1 - \frac{16\alpha_2^2}{\sqrt{\omega^2}} + \frac{27m_2^2}{\omega^2}},
\]

\[
\alpha_4 = \sqrt[3]{\frac{18}{\omega^2}} \sqrt[3]{\alpha_2 m_2^2 - 12^{2/3} \sqrt[3]{\alpha_2} m_2^2 (27m_2^2 + 256) + m_2^2 (16\sqrt[2]{2^{2/3} \alpha_2^{2/3} - 3\sqrt{81m_2^8 + 768m_2^6}} + 27m_2^2)},
\]

\[
\alpha_5 = \sqrt{\frac{3\alpha_2 m_2^2 - 9\sqrt[3]{2m_2^2 - \sqrt[3]{81m_2^8 + 768m_2^6}}}{\alpha_2^{2/3} m_2^2}},
\]

the coefficients are given by

\[
d_{4000} = \frac{4(m_2 - 4\omega_2)^2 (m_1^2 - m_1\omega_2)^2 - 4096\sqrt[3]{3}^{2/3} \alpha_3 \left( 2^{2/3} \sqrt[3]{3} \alpha_2^{2/3} + \left( 3 \sqrt[3]{\alpha_2} - 8\sqrt[3]{23}^{2/3} \right) m_2^2 \right) (m_1^2 - m_1\omega_2)^4}{\sqrt[3]{\alpha_2^2} m_1^2 \omega_2^2 \left( (8m_2^2 - 3m_2) \sqrt{\alpha_1 m_2 + \alpha_1 m_2} \right)^2 \left( 8\frac{\sqrt[3]{6m_2^2 - 2^{2/3} \alpha_2^{2/3}}}{\sqrt[3]{\alpha_2^2} m_2^2} \right)^{5/2}},
\]

\[
d_{0400} = \frac{4m_1^2 \left( -4m_1^2 + m_2 + 2\omega_2 \right)^2 \left( -4m_1^2 + m_2 + 4\omega_2 \right)^2}{\omega_2^2 \left( \alpha_1 m_2 - (8m_2^2 - 3m_2) \sqrt{\alpha_1 m_2} \right)^2}

- \frac{16\sqrt[3]{3}^{2/3} \alpha_3 m_1^4 \left( 2^{2/3} \sqrt[3]{3} \alpha_2^{2/3} + \left( 3 \sqrt[3]{\alpha_2} - 8\sqrt[3]{23}^{2/3} \right) m_2^2 \right) \left( -4m_1^2 + m_2 + 4\omega_2 \right)^4}{\sqrt{\alpha_2^2} m_1^2 \omega_2^2 \left( \alpha_1 m_2 - (8m_2^2 - 3m_2) \sqrt{\alpha_1 m_2} \right)^2 \left( \frac{8\sqrt[3]{6m_2^2 - 2^{2/3} \alpha_2^{2/3}}}{\sqrt[3]{\alpha_2^2} m_2^2} \right)^{5/2}}
\]

\[
d_{0040} = \frac{4m_1^2 \omega_2^2 \left( 4m_1^2 + m_2 - 4\omega_2 \right)^2}{\left( \frac{1}{4} \left( 8\omega_2^2 - 2m_2 \right)^2 - 16m_1^4 + 4m_2 m_1^2 \right)^2}.
\]
\[
2\sqrt{3}^{2/3} \alpha_3 \omega_1^2 \left(4 \cdot 2^{2/3} \sqrt[3]{\alpha_2}^{2/3} + \left(9 \sqrt{\alpha_2} - 32 \sqrt[3]{23}^{2/3}\right) m_1^3 \right) \left(4 m_1^2 + m_2 - 4 \omega_2^2 \right)^4 \\
\sqrt{\alpha_2} m_2^2 \left(8 \cdot 4^{2/3} \alpha_2 \omega_2^2 \right) \left((8 m_1^2 - 3 m_2) \sqrt{\alpha_1 m_2^2 + \alpha_1 m_2} \right)^2 \left(\frac{8 \sqrt{3} m_2^2 - 2^{2/3} \alpha_3 \omega_2^2}{\sqrt{\alpha_1 m_2^2 + \alpha_1 m_2}}\right)^{5/2}.
\]

\[
d_{0004} = \frac{64 m_1^2 \omega_2^2 (-2 m_1^2 + m_2 + 2 \omega_2^2)^2}{\left(8 \cdot 2^{2/3} \sqrt[3]{\alpha_2}^{2/3} + \left(9 \sqrt{\alpha_2} - 32 \sqrt[3]{23}^{2/3}\right) m_1^3 \right) \left(4 m_1^2 + m_2 - 4 \omega_2^2 \right)^4} \\
\sqrt{\alpha_2} m_2^2 \left(8 \cdot 4^{2/3} \alpha_2 \omega_2^2 \right) \left((8 m_1^2 - 3 m_2) \sqrt{\alpha_1 m_2^2 + \alpha_1 m_2} \right)^2 \left(\frac{8 \sqrt{3} m_2^2 - 2^{2/3} \alpha_3 \omega_2^2}{\sqrt{\alpha_1 m_2^2 + \alpha_1 m_2}}\right)^{5/2}.
\]

\[
d_{2200} = \left[\frac{1}{\sqrt[3]{\alpha_2}^{5/2}} \left(\frac{1}{\sqrt{\alpha_2}^{2/3}} - \frac{1}{\sqrt{\alpha_2}^{2/3}} \alpha_2 \right) \left(8 \cdot 2^{2/3} \sqrt[3]{\alpha_2}^{2/3} + \left(9 \sqrt{\alpha_2} - 32 \sqrt[3]{23}^{2/3}\right) m_1^3 \right) \left(4 m_1^2 + m_2 - 4 \omega_2^2 \right)^4 \right] \\
\left(16 m_1^5 \omega_2 (128 m_1^4 - 68 m_2 m_1^2 + 9 m_2^2) \omega_1 \omega_2 \right).
\]

\[
d_{2020} = \frac{128 \sqrt{3}^{2/3} \alpha_3 \left(7 \cdot 2^{2/3} \sqrt[3]{\alpha_2}^{2/3} + \left(18 \sqrt{\alpha_2} - 56 \sqrt[3]{23}^{2/3}\right) m_1^3 \right) \left(4 m_1^2 + m_2 - 4 \omega_2^2 \right)^2 \left(9 m_1^3 - m_1 \omega_2^2 \right)^2 + \\
-512 m_1^6 \omega_2^2 + 16 m_1^4 \left(-8 m_2 \omega_2^2 + m_2^2 + 32 \omega_2^4 \right) + 8 m_1^2 \left(m_2 - 4 \omega_2^2 \right)^3 + \left(m_2 - 4 \omega_2^2 \right)^4 + 256 m_1^8}{4 \left(\left(4 \cdot 2^{2/3} \sqrt[3]{\alpha_2}^{2/3} + \left(9 \sqrt{\alpha_2} - 32 \sqrt[3]{23}^{2/3}\right) m_1^3 \right) \left(4 m_1^2 + m_2 - 4 \omega_2^2 \right)^4 \right)^2} \\
\left(16 m_1^5 \omega_2 (128 m_1^4 - 68 m_2 m_1^2 + 9 m_2^2) \omega_1 \omega_2 \right).
\]

\[
d_{0202} = \frac{1}{\sqrt[3]{\alpha_2}^{5/2}} \left(\frac{1}{\sqrt{\alpha_2}^{2/3}} - \frac{1}{\sqrt{\alpha_2}^{2/3}} \alpha_2 \right) \left(8 \cdot 2^{2/3} \sqrt[3]{\alpha_2}^{2/3} + \left(9 \sqrt{\alpha_2} - 32 \sqrt[3]{23}^{2/3}\right) m_1^3 \right) \left(4 m_1^2 + m_2 - 4 \omega_2^2 \right)^4 \\
\left(32 \sqrt{3}^{2/3} \alpha_3 \omega_2 \left(7 \cdot 2^{2/3} \sqrt[3]{\alpha_2}^{2/3} + \left(18 \sqrt{\alpha_2} - 56 \sqrt[3]{23}^{2/3}\right) m_1^3 \right) \left(4 m_1^2 + m_2 - 4 \omega_2^2 \right)^4 \right)^2 \\
\left(\left(-2 m_1^2 + m_2 + 2 \omega_2^2 \right)^2 \right) - \frac{1}{4 m_1^2 m_2 (128 m_1^4 - 68 m_2 m_1^2 + 9 m_2^2) \omega_1} \\
\left(\omega_2 \left(-128 m_1^6 \omega_2^2 + 4 m_1^4 \left(-8 m_2 \omega_2^2 + m_2^2 + 32 \omega_2^4 \right) - 4 m_1^2 \left(m_2 - 4 \omega_2^2 \right)^2 \left(m_2 - 4 \omega_2^2 \right)^4 \right) \left(-2 m_2 \omega_2^2 + m_2^2 - 8 \omega_2^4 \right)^2 + 64 m_1^8 \right) \\
\left(\left(-2 m_1^2 + m_2 + 2 \omega_2^2 \right)^2 \right).
\[
  d_{0220} = \frac{\omega_1 \left( (-4m_1^2 + m_2 + 2\omega_2^2)^2 (-4m_1^2 + m_2 + 4\omega_2^2)^2 - 4m_1^4 (-4m_1^2 + m_2 + 4\omega_2^2)^2 \right)}{4m_1^2m_2 (128m_1^4 - 68m_2m_1^2 + 9m_2^2) \omega_2}
  \\
  \times 3^{2/3} \alpha_3 \omega_1 \left( 18 \sqrt{\frac{3\omega_2}{2}} - 56 \sqrt{23}/\sqrt{3} \right) m_2^4 (4m_1^2 + m_2 - 4\omega_2^2)^2.
\]

\[
  d_{0202} = \frac{\sqrt{2} \sqrt{\frac{3\alpha_2}{2}} m_2^5 (128m_1^4 - 68m_2m_1^2 + 9m_2^2) \omega_2 \left( \frac{8 \sqrt{6\omega_2^2 - 2^3/2^3 \alpha_3^2}}{\sqrt{\alpha_2} m_2^2} \right)^{5/2}}{\left(-4m_1^2 + m_2 + 4\omega_2^2\right)^2},
\]

\[
  d_{0022} = \left[-4m_2^4 \left(-2m_1^2 + m_2 + 2\omega_2^2\right)^2 - 8m_2^2 \left(4m_1^2 + m_2 - 4\omega_2^2\right) \left(-2m_1^2 + m_2 + 2\omega_2^2\right) + 3 \sqrt{3} \alpha_4 \left(4 \sqrt{2^2/3 \sqrt{3\omega_2^2}} + \left(9 \sqrt{\omega_2} - 32 \sqrt{23}/\sqrt{3} \right) m_2^2 \right) \left(4m_1^2 + m_2 - 4\omega_2^2\right) \right]
  \times \frac{\omega_1 \omega_2}{m_2^5 (128m_1^4 - 68m_2m_1^2 + 9m_2^2)}.
\]

**Appendix B. Coefficient for \(H_2\) at the south pole**

We define the following quantities

\[
  \beta_1 = \sqrt{m_2 \left(32m_1^2 + 9m_2\right) (8m_1^2 + 3m_2) + m_2 \left(32m_1^2 + 9m_2\right)},
\]

\[
  \beta_2 = \sqrt{81m_1^8 + 768m_2^6 - 9m_4^4},
\]

\[
  \beta_3 = \sqrt{-\frac{16 \sqrt{7}}{3 \beta_5} + \frac{\beta_3}{m_2^2} + 1},
\]

\[
  \beta_4 = (8m_1^2 + 3m_2) \sqrt{m_2 \left(32m_1^2 + 9m_2\right) - m_2 \left(32m_1^2 + 9m_2\right)},
\]

\[
  \beta_5 = 9m_2^6 - m_1^2 \sqrt{81m_1^8 + 768m_2^6},
\]

\[
  \beta_6 = \frac{1}{\beta_2} \left(18 \sqrt{2^3/3 \sqrt{3\beta_5^2} m_2^4} - 2 \sqrt{2^2/3 \beta_5^2} \sqrt{m_2^7 (27m_2^2 + 256) + m_2^5} \left(16 \sqrt{23}/\sqrt{3}\beta_2^2 - 3 \sqrt{81m_1^8 + 768m_2^6} + 27m_2^4\right)\right).
\]

with \(m_1, m_2\) defined in (68) and \(\beta_6 > 0\), then the coefficients in this case are given by
\[d_{4000} = \frac{32768 \, 3^{2/3} \sqrt{\beta_2 \beta_3 m_2} \left(2^{2/3} \sqrt{3} \beta_2^{2/3} + \left(3 \sqrt{\beta_2} - 8 \sqrt[3]{23^{2/3}} \right) m_2^2 \right) \left(m_1^3 - m_1 \omega_2^2 \right)^4 \right)}{\beta_1^2 \omega_1^2 \left(16 \sqrt[3]{6} m_2^2 - 2 \, 2^{2/3} \beta_2^{2/3} \right)^{5/2}} + \]

\[16 \left(m_2 + 4 \omega_2^2 \right)^2 \left(m_1^3 - m_1 \omega_2^2 \right)^2 \]

\[d_{0400} = \frac{128 \, 3^{2/3} \sqrt{\beta_2 \beta_3 m_1^4 m_2} \left(2^{2/3} \sqrt{3} \beta_2^{2/3} + \left(3 \sqrt{\beta_2} - 8 \sqrt[3]{23^{2/3}} \right) m_2^2 \right) \left(4m_1^2 + m_2 - 4 \omega_2^2 \right)^4 \right)}{\beta_1^2 \omega_1^2 \left(16 \sqrt[3]{6} m_2^2 - 2 \, 2^{2/3} \beta_2^{2/3} \right)^{5/2}} + \]

\[\frac{4m_1^2 \left(4m_1^2 + m_2 - 2 \omega_2^2 \right)^2 \left(4m_1^2 + m_2 - 4 \omega_2^2 \right)^2 \right)}{\beta_1^2 \omega_1^2} \]

\[d_{0040} = \frac{16 \, 3^{2/3} \sqrt{\beta_2 \beta_3 m_2} \omega_1^2 \left(4 \, 2^{2/3} \sqrt{3} \beta_2^{2/3} + \left(9 \, 3 \beta_2 - 32 \sqrt[3]{23^{2/3}} \right) m_2^2 \right) \left(-4m_1^2 + m_2 + 4 \omega_2^2 \right)^4 \right)}{\beta_1^2 \left(16 \sqrt[3]{6} m_2^2 - 2 \, 2^{2/3} \beta_2^{2/3} \right)^{5/2}} + \]

\[\frac{16m_1^2 \omega_1^2 \left(-4m_1^2 + m_2 + 4 \omega_2^2 \right)^2 \right)}{\beta_1^2 \omega_1^2} \]

\[d_{0004} = \frac{256 \, 3^{2/3} \sqrt{\beta_2 \beta_3 m_2} \omega_1^2 \left(4 \, 2^{2/3} \sqrt{3} \beta_2^{2/3} + \left(9 \sqrt[3]{2} \sqrt[3]{3} \beta_2 - 32 \sqrt[3]{23^{2/3}} \right) m_2^2 \right) \left(2m_1^2 + m_2 - 2 \omega_2^2 \right)^4 \right)}{\beta_1^2 \left(16 \sqrt[3]{6} m_2^2 - 2 \, 2^{2/3} \beta_2^{2/3} \right)^{5/2}} + \]

\[+ \frac{64m_1^2 \omega_1^2 \left(2m_1^2 + m_2 - 2 \omega_2^2 \right)^2 \right)}{\beta_1^2 \omega_1^2} \]

\[d_{3100} = \frac{8m_2^2 \omega_1^{3/2} \sqrt{\omega_2} \left(\beta_1 m_2 \right)^{3/2} \sqrt{\beta_4 m_2} \right)}{\omega_1^{3/2} \sqrt{\omega_2} \left(\beta_1 m_2 \right)^{3/2} \sqrt{\beta_4 m_2} \right)} \left[m_1^2 \left(m_1^3 - \omega_2^2 \right) \left(m_2 + 4 \omega_2^2 \right)^2 \left(4m_1^2 + m_2 - 4 \omega_2^2 \right)^2 \right) - \]

\[8 \left(4m_1^2 + m_2 - 2 \omega_2^2 \right) \left(m_1^3 - m_1 \omega_2^2 \right)^2 \right) - \]

\[256 \sqrt[3]{2} \sqrt[3]{3} \sqrt{\beta_2 \beta_3 m_2} \left(\left(8 \, 6^{2/3} - 3 \sqrt[3]{2} \sqrt[3]{3} \beta_2 \right) m_2^2 - 2 \sqrt[3]{3} \beta_2^{2/3} \right) \left(4m_1^2 + m_2 - 4 \omega_2^2 \right) \left(m_1^3 - \omega_2^2 \right)^2 \right) \]

\[\left(8 \sqrt[3]{3} m_2^2 - \sqrt[3]{2} \beta_2^{2/3} \right)^{5/2} \]

\[d_{1300} = \frac{4m_1^2 \omega_1^{3/2} \sqrt{\omega_2} \left(\beta_1 m_2 \right)^{3/2} \sqrt{\beta_4 m_2} \right)}{\omega_1^{3/2} \sqrt{\omega_2} \left(\beta_1 m_2 \right)^{3/2} \sqrt{\beta_4 m_2} \right)} \left[8 \left(m_1^3 - \omega_2^2 \right) \left(4m_1^2 + m_2 - 2 \omega_2^2 \right)^2 \right) - \]

\[\left(4m_1^2 + m_2 - 2 \omega_2^2 \right) \left(m_1^3 - m_1 \omega_2^2 \right)^2 \right) - \]

\[32 \sqrt[3]{3} \sqrt[3]{3} \sqrt{\beta_2 \beta_3 m_2} \left(\left(8 \, 6^{2/3} - 3 \sqrt[3]{2} \sqrt[3]{3} \beta_2 \right) m_2^2 - 2 \sqrt[3]{3} \beta_2^{2/3} \right) \left(4m_1^2 + m_2 - 4 \omega_2^2 \right)^2 \left(m_1^3 - \omega_2^2 \right)^2 \right) \]

\[\left(8 \sqrt[3]{3} m_2^2 - \sqrt[3]{2} \beta_2^{2/3} \right)^{5/2} \]

\[d_{0031} = \frac{8m_2^2 \omega_1^{3/2} \sqrt{\omega_2} \left(-4m_1^2 + m_2 + 4 \omega_2^2 \right)}{(\beta_1 m_2)^{3/2} \sqrt{\beta_4 m_2} \right)} \left[4m_1^2 \left(-4m_1^2 + m_2 + 4 \omega_2^2 \right)^2 \right) + \]

\[8m_1^2 \left(2m_1^2 + m_2 - 2 \omega_2^2 \right) \]
\[
\begin{align*}
&-\frac{\sqrt[3]{3}\sqrt[3]{\beta_6 m_2} \left( \left( 32 \, 2^{6/3} - 9 \, 3^{2} \sqrt{3} \beta_6^2 \right) m_2^2 - 8 \, \sqrt{3} \beta_6^{2/3} \right) \left( (2m_1^2 + m_2 - 2\omega_2^2) \right) \left( -4m_1^2 + m_2 + 4\omega_2^2 \right)^2}{\left( 8 \, \sqrt{3} m_2^2 - \sqrt[3]{2} \beta_6^{2/3} \right)^{5/2}} \\
d_{0013} &= \frac{32m_2^2 \sqrt[3]{\omega_1 \omega_2} \beta_6 m_2 \left( 2m_1^2 + m_2 - 2\omega_2^2 \right)}{\beta_1 m_2 (\beta_4 m_2)^{3/2}} \left[ 2m_1^2 \left( -4m_1^2 + m_2 + 4\omega_2^2 \right) \right. \\
&\quad + 4m_1^2 \left( 2m_1^2 + m_2 - 2\omega_2^2 \right) \\
&\quad - \frac{\sqrt[3]{3}\sqrt[3]{\beta_6} m_2 \left( \left( 32 \, 2^{6/3} - 9 \, 3^{2} \sqrt{3} \beta_6^2 \right) m_2^2 - 8 \, \sqrt{3} \beta_6^{2/3} \right) \left( 2m_1^2 + m_2 - 2\omega_2^2 \right) \left( -4m_1^2 + m_2 + 4\omega_2^2 \right)^2}{\left( 8 \, \sqrt{3} m_2^2 - \sqrt[3]{2} \beta_6^{2/3} \right)^{5/2}} \\
d_{2200} &= \frac{1}{16 \left( 128m_1^4 + 68m_2 m_1^2 + 9m_2^2 \right) \omega_1 \omega_2} \left[ \frac{(m_2 + 4\omega_2^2)^2 \left( 4m_1^2 + m_2 - 2\omega_2^2 \right)^2}{m_2} \right. \\
&\quad + \frac{768 \, 2^{2/3} \sqrt[3]{3} \sqrt[3]{\beta_6} m_2 \left( 2^{2/3} \sqrt[3]{3} \beta_6^{2/3} \left( 3 \, 2^{6/3} - 9 \sqrt{3} \beta_6^{2/3} \right) m_2^2 - 8 \, \sqrt{3} \beta_6^{2/3} \right) \left( m_1^2 - \omega_2^2 \right)^2 \left( 4m_1^2 + m_2 - 2\omega_2^2 \right)^2}{\left( 8 \, \sqrt{3} m_2^2 - \sqrt[3]{2} \beta_6^{2/3} \right)^{5/2}} \\
&\quad - \frac{32 \left( m_1^2 - \omega_2^2 \right) \left( 128m_1^4 \omega_2^2 + 2m_1^2 \left( 16m_2 \omega_2^2 + 3m_2^2 - 68 \omega_2^4 \right) \right) - 24m_2 \omega_2^4 - 32m_1^4 + m_3^2 + 40\omega_2^6}{m_2} \\
d_{2020} &= \frac{1}{4} \left[ \frac{1024m_1^2 \left( m_1^3 - m_1 \omega_2^2 \right)^2 \beta_1^2}{\beta_1^2} \right. \\
&\quad + \frac{4 \left( m_2 + 4\omega_2^2 \right)^2 \left( -4m_1^2 + m_2 + 4\omega_2^2 \right)^2 \beta_1^2}{\beta_1^2} \\
&\quad - \frac{4096 \, 3^{2/3} \sqrt[3]{\beta_6} m_2 \left( 7 \, 2^{2/3} \sqrt[3]{3} \beta_6^{2/3} + \left( 18 \, \sqrt[3]{\beta_6} - 56 \, \sqrt[3]{3} \beta_6^{2/3} \right) m_2^2 \right) \left( m_1^2 - \omega_2^2 \right)^2 \left( -4m_1^2 + m_2 + 4\omega_2^2 \right)^2}{\beta_1^2 \left( 16 \sqrt[3]{m_2^2} - 2 \, 2^{2/3} \beta_6^{2/3} \right)^{5/2}} \\
d_{0202} &= \frac{\omega_2}{4 \left( 128m_1^4 + 68m_2 m_1^2 + 9m_2^2 \right) \omega_1} \left[ \frac{(m_2 + 4\omega_2^2)^2 \left( 2m_1^2 + m_2 - 2\omega_2^2 \right)^2}{m_1^2 m_2} \right. \\
&\quad + \frac{64 \left( m_3^2 - m_1 \omega_2^2 \right)^2}{m_2} \\
&\quad - \frac{64 \, 2^{2/3} \sqrt[3]{\beta_6} m_2 \left( 7 \, 2^{2/3} \sqrt[3]{3} \beta_6^{2/3} + \left( 18 \, \sqrt[3]{\beta_6} - 56 \, \sqrt[3]{3} \beta_6^{2/3} \right) m_2^2 \right) \left( m_1^2 - \omega_2^2 \right)^2 \left( 2m_1^2 + m_2 - 2\omega_2^2 \right)^2}{\left( 8 \, \sqrt{3} m_2^2 - \sqrt[3]{2} \beta_6^{2/3} \right)^{5/2}} \\
d_{0020} &= \frac{\omega_1}{4 \left( 128m_1^4 + 68m_2 m_1^2 + 9m_2^2 \right) \omega_2} \left[ \frac{( -4m_1^2 + m_2 + 4\omega_2^2)^2 \left( 4m_1^2 + m_2 - 2\omega_2^2 \right)^2}{m_1^2 m_2} \right. \\
&\quad + \frac{4m_1^2 \left( 4m_1^2 + m_2 - 4\omega_2^2 \right)^2}{m_2} \\
&\quad - \frac{2^{2/3} \sqrt[3]{\beta_6} m_2 \left( 7 \, 2^{2/3} \sqrt[3]{3} \beta_6^{2/3} + \left( 18 \, \sqrt[3]{\beta_6} - 56 \, \sqrt[3]{3} \beta_6^{2/3} \right) m_2^2 \right) \left( 4m_1^2 + m_2 - 4\omega_2^2 \right)^2 \left( -4m_1^2 + m_2 + 4\omega_2^2 \right)^2}{\left( 8 \, \sqrt{3} m_2^2 - \sqrt[3]{2} \beta_6^{2/3} \right)^{5/2}} \\
&\right] 
\end{align*}
\]
\[
\begin{align*}
d_{0202} &= \frac{64m_1^4 (4m_1^2 + m_2 - 4\omega_2^2)^2 + 64 (2m_1^2 + m_2 - 2\omega_2^2)^2 (4m_1^2 + m_2 - 2\omega_2^2)^2}{4\beta_2^4} \\
&- \frac{1024 \sqrt[2]{3/3} \beta_2 \beta_3 m_1^2 m_2 \left( 7 \sqrt[2]{3/3} \beta_2^{3/3} + \left( 18 \sqrt[2]{3/3} - 56 \sqrt[2]{3} \beta_2^{3/3} \right) m_2^2 \right) (4m_1^2 + m_2 - 4\omega_2^2)^2 (2m_1^2 + m_2 - 2\omega_2^2)^2}{4\beta_2^4 \left( 16\sqrt[2]{6} m_2^4 - 2 \sqrt[2]{3} \beta_2^{3/3} \right)^{5/2}} \\

\begin{align*}
d_{0022} &= \frac{\omega_1 \omega_2}{2 \left( 128 m_1^4 + 68m_2 m_1^2 + 9m_2^2 \right)} \left[ 2 \left( 64m_1^4 \omega_2^2 + 8m_2 \omega_2^2 - 32m_1^4 - 8m_2 m_1^2 + 13m_2^2 - 32\omega_2^4 \right) \right] \\
&+ \frac{3 \sqrt[2]{3/3} \beta_2 \beta_6 m_2 \left( 4 \sqrt[2]{3} \beta_2^{2/3} + \left( 9 \sqrt[2]{3} - 32 \sqrt[2]{3} \beta_2^{3/3} \right) m_2^2 \right)}{m_1^2 \left( 8 \sqrt[3]{3} m_2^2 - \sqrt[2]{2} \beta_2^{3/3} \right)^{5/2}} \\
&\left( -4m_1^2 + m_2 + 4\omega_2^2 \right)^2 \left( 2m_1^2 + m_2 - 2\omega_2^2 \right)^2 \\

\begin{align*}
d_{2011} &= \frac{4m_2^2 \sqrt[2]{\omega_2 / \omega_1}}{(\beta_1 m_2)^{3/2} \sqrt[2]{3} \beta_4 m_2} \left( 128 m_1^4 \left( m_1^2 - \omega_2^2 \right)^2 + \left( m_2 + 4\omega_2^2 \right)^2 \left( -4m_1^2 + m_2 + 4\omega_2^2 \right) \right) \\
&\left( 2m_1^2 + m_2 - 2\omega_2^2 \right) + \frac{128 \sqrt[2]{2 \sqrt[3]{3} \beta_2 \beta_6 m_2 \left( \left( 28 \sqrt[2]{3} - 9 \sqrt[2]{2} \sqrt[2]{3} \beta_2^{3/3} \right) m_2^2 - 7 \sqrt[3]{3} \beta_2^{3/3} \right) (-4m_1^2 + m_2 + 4\omega_2^2)}{\left( 8 \sqrt[3]{3} m_2^2 - \sqrt[2]{2} \beta_2^{3/3} \right)^{5/2}} \\
&\left( 2m_1^2 + m_2 - 2\omega_2^2 \right) \left( m_1^2 - m_1 \omega_2^2 \right)^2 \\

\begin{align*}
d_{0211} &= \frac{16m_2^2 \sqrt[2]{\omega_1 / \omega_2}}{\sqrt[2]{\beta_1 m_2} (\beta_4 m_2)^{3/2}} \left( 2m_1^4 \left( 4m_1^2 + m_2 - 4\omega_2^2 \right)^2 + \left( -4m_1^2 + m_2 + 4\omega_2^2 \right) \right) \cdot \\
&\left( 2m_1^2 + m_2 - 2\omega_2^2 \right) \left( 4m_1^2 + m_2 - 2\omega_2^2 \right)^2 + \frac{2 \sqrt[2]{2 \sqrt[3]{3} \beta_2 \beta_6 m_2 \left( \left( 28 \sqrt[2]{3} - 9 \sqrt[2]{2} \sqrt[2]{3} \beta_2^{3/3} \right) m_2^2 - 7 \sqrt[3]{3} \beta_2^{3/3} \right) (-4m_1^2 + m_2 + 4\omega_2^2)}{\left( 8 \sqrt[3]{3} m_2^2 - \sqrt[2]{2} \beta_2^{3/3} \right)^{5/2}} \\
&\left( 2m_1^2 + m_2 - 2\omega_2^2 \right) \left( 4m_1^2 + m_2 - 4\omega_2^2 \right)^2 \\

\begin{align*}
d_{1120} &= \frac{4m_2^2 \sqrt[2]{\omega_1 / \omega_2}}{(\beta_1 m_2)^{3/2} \sqrt[2]{3} \beta_4 m_2} \left( 32m_1^4 \left( m_1^2 - \omega_2^2 \right) \left( 4m_1^2 + m_2 - 4\omega_2^2 \right) - \left( 4m_1^2 + m_2 - 2\omega_2^2 \right) \right) \cdot \\
&\left( m_2 + 4\omega_2^2 \right) \left( -4m_1^2 + m_2 + 4\omega_2^2 \right)^2 + \frac{16 \sqrt[2]{2 \sqrt[3]{3} \beta_2 \beta_6 m_2 m_1 \left( \left( 28 \sqrt[2]{3} - 9 \sqrt[2]{2} \sqrt[2]{3} \beta_2^{3/3} \right) m_2^2 - 7 \sqrt[3]{3} \beta_2^{3/3} \right) \left( m_1^2 - \omega_2^2 \right)}{\left( 8 \sqrt[3]{3} m_2^2 - \sqrt[2]{2} \beta_2^{3/3} \right)^{5/2}} \right) \\
&\left( -4m_1^2 + m_2 + 4\omega_2^2 \right)^2 \left( 4m_1^2 + m_2 - 4\omega_2^2 \right) \\
\end{align*}
\end{align*}
\end{align*}
\]
\[ d_{1102} = \frac{16m_2^4 \sqrt{\frac{\omega_2}{\omega_1}}}{\sqrt{\beta_1 m_2 (\beta_4 m_2)}^{3/2}} \left( 8m_1^4 \left( m_1^2 - \omega_2^2 \right) \left( 4m_1^2 + m_2 - 2\omega_2^2 \right) - \left( 2m_1^2 + m_2 - 2\omega_2^2 \right)^2 \right). \]

\[ \left( 4m_1^2 + m_2 - 2\omega_2^2 \right) \left( m_2 + 4\omega_2^2 \right) + \]

\[ 16\sqrt{3} \sqrt{\beta_3 \beta_6} m_2 m_1 \left( \left( 28 \frac{6^{2/3}}{3} \sqrt[3]{2} \beta_2^2 - 7 \frac{\sqrt{2} \beta_2^{2/3}}{3} \right) m_2^2 \right) \]

\[ \left( \frac{8 \sqrt{3} m_2^2 - \frac{\sqrt{2} \beta_2^{2/3}}{3} }{2} \right)^{5/2} \]

\[ \left( m_1^2 - \omega_2^2 \right) \left( 2m_1^2 + m_2 - 2\omega_2^2 \right)^2 \left( 4m_1^2 + m_2 - 4\omega_2^2 \right) \]

\[ d_{1111} = \frac{1}{128m_1^4 + 68m_2 m_1^2 + 9m_2^2} \left[ \frac{16m_1^4 \left( m_1^2 - \omega_2^2 \right) \left( 4m_1^2 + m_2 - 4\omega_2^2 \right)}{m_2} - \left( 2m_1^2 + m_2 - 2\omega_2^2 \right) \left( 4m_1^2 + m_2 - 2\omega_2^2 \right) \left( m_2 + 4\omega_2^2 \right) \left( -4m_1^2 + m_2 + 4\omega_2^2 \right) \right] \]

\[ 8 \left( 2^{2/3} \sqrt[3]{3} \beta_2^2 + \left( 18 \sqrt[3]{2} \beta_2 - 56 \sqrt{2} \beta_2^{2/3} \right) m_2^2 \right) \left( 4m_1^2 + m_2 - 4\omega_2^2 \right) \]

\[ \left( \frac{8 \sqrt{3} m_2^2 - \frac{\sqrt{2} \beta_2^{2/3}}{3} }{2} \right)^{5/2} \]

\[ \left( 2m_1^2 + m_2 - 2\omega_2^2 \right) \left( m_1^2 - \omega_2^2 \right) \left( -4m_1^2 + m_2 + 4\omega_2^2 \right) \].

References


